Published by Institute of Physics Publishing for SISSA

Received: *August 30, 2007*
Revised: *November 15, 2007*
Accepted: *December 8, 2007*
Published: *December 12, 2007*

Emergent gravity from noncommutative gauge theory

Harold Steinacker

Fakultät für Physik, Universität Wien,
Boltzmanngasse 5, A-1090 Wien, Austria
E-mail: `harold.steinacker@univie.ac.at`

Abstract: We show that the matrix-model action for noncommutative $\mathrm{U}(n)$ gauge theory actually describes $\mathrm{SU}(n)$ gauge theory coupled to gravity. This is elaborated in the 4-dimensional case. The $\mathrm{SU}(n)$ gauge fields as well as additional scalar fields couple to an effective metric G_{ab}, which is determined by a dynamical Poisson structure. The emergent gravity is intimately related to noncommutativity, encoding those degrees of freedom which are usually interpreted as $\mathrm{U}(1)$ gauge fields. This leads to a class of metrics which contains the physical degrees of freedom of gravitational waves, and allows to recover e.g. the Newtonian limit with arbitrary mass distribution. It also suggests a consistent picture of UV/IR mixing in terms of an induced gravity action. This should provide a suitable framework for quantizing gravity.

Keywords: Gauge Symmetry, M(atrix) Theories, Models of Quantum Gravity, Non-Commutative Geometry.

Contents

1. Introduction

It is generally accepted that the classical concepts of space and time will break down at the Planck scale, where quantum fluctuations of space-time due to the interplay between gravity and quantum mechanics become important. One way to approach this problem is to replace classical space-time by some kind of quantum space, incorporating space-time uncertainty relations such as those obtained in [1]. This leads to noncommutative (NC) field theory, where some fixed NC space is assumed; for basic reviews see e.g. [2].

After considerable progress in the understanding of field theory on "fixed" NC or quantum spaces, it is of fundamental importance to understand how a dynamical quantum space in the spirit of general relativity can be incorporated in such a framework. If noncommutative spaces are related to quantum gravity, the incorporation of gravity should be simple and natural. Furthermore, one should take into account the lessons from string theory, which provides a realization of quantum spaces as D-branes in a nontrivial B-field background [3], and points to a relation with gravity [4 − 11]. While several formulations of NC gravity have been proposed by deforming various formulations of general relativity [17 − 25], a simple and compelling mechanism would be very desirable.

JHEP12(2007)049

To identify this mechanism, it is helpful to reconsider gauge theories. There is a very simple and natural formulation of $\mathfrak{u}(n)$ NC gauge theory in terms of matrix models, typically of the form $S \cong Tr[X^a, X^b][X^a, X^b] + \cdots$. Such actions describes gauge theory on the quantum plane $\mathbb{R}^d_\theta$. Similar actions arise in the context of string theory, such as the IKKT model [7]. The dynamical objects are matrices resp. operators $X^a = Y^a + A^a \in \mathcal{A} \otimes \mathfrak{u}(n)$ ("covariant coordinates"), where Y^a generates the algebra of functions $\mathcal{A} \cong L(\mathcal{H})$ on some NC space. The central observation is that the fluctuations $A^a \in \mathcal{A}$ of the covariant coordinates can be interpreted as $\mathfrak{u}(n)$-valued gauge fields on the NC space. These considerations become more rigorous for compact quantum spaces such as $\mathbb{C}P^2_N$ or $S^2_N \times S^2_N$, which are described by *finite* matrix models of similar type [26, 27].

Even though this realization of gauge fields is very appealing, it is nevertheless strange: fluctuations of NC coordinates ought to describe fluctuations of the geometry, rather than gauge fields. This is particularly compelling for gauge theory "on" fuzzy spaces such as $\mathbb{C}P^2_N$ or $S^2_N \times S^2_N$, where the geometry of the space is indeed dynamical and given by the minimum $\langle X^a \rangle = Y^a$ of an appropriate matrix model. This strongly hints at an implicit gravity sector. There is also strong evidence for the presence of gravity in the IKKT matrix model of type IIB string theory [7 − 11], and even for a D=4 compactification thereof [9, 11] which can be viewed as a supersymmetric version of the model which will be studied here. Further striking parallels between gravity and NC gauge theory include the absence of local observables, and the implementation of translations as gauge transformations. Finally, the $\mathfrak{u}(1)$ sector of D=4 noncommutative gauge theory is afflicted by the infamous UV/IR mixing [28 − 30], leading to a behavior which is very different from electrodynamics.

We show in this paper that the matrix model formulation of NC gauge theory in 4 dimensions does in fact contain gravity. More precisely, it should be interpreted as $\mathfrak{su}(n)$ gauge theory coupled to gravity, with dynamical geometry determined by $\mathfrak{u}(1)$ components of the covariant coordinates X^a. This solves at the same time a long-standing problem how to define NC $\mathfrak{su}(n)$ gauge theory: It has been known that the $\mathfrak{u}(1)$ sector of NC gauge theory cannot be disentangled from the $\mathfrak{su}(n)$ sector in any obvious way. Here we understand this fact as the coupling of the $\mathfrak{su}(n)$ gauge fields to gravity.

One may wonder how it is possible that nontrivial geometries arise from what is usually interpreted as $\mathfrak{u}(1)$ gauge fields. The answer is quite simple: the effective geometry is determined by the metric $G^{ab} = -\theta^{ac}(y)\theta^{bd}(y)g_{cd}$, where $\theta^{ac}(y) = \overline{\theta}^{ac} + F^{ab}(y)$ is the dynamical Poisson tensor which is usually split into background NC space and $\mathfrak{u}(1)$ field strength, for $g_{ab} = \delta_{ab}$ resp. $g_{ab} = \eta_{ab}$ in the Euclidean resp. Minkowski case. While such metrics do not reproduce the most general geometries, they do contain the physical degrees of freedom of gravitational waves, and allow to obtain e.g. the Newtonian limit. Therefore this provide a physically viable class of geometries for gravity.

The observation that gravity can arises from NC gauge theory is not new. In particular, Rivelles [31] found a linearized version of the same effective metric coupling to scalar fields on $\mathbb{R}^4_\theta$, without addressing however nonabelian gauge fields. The idea that NC $\mathfrak{u}(1)$ gauge theory should be viewed as gravity was put forward explicitly in [32, 33] from the string theory point of view. We establish this mechanism in detail based on a very simple and explicit matrix model, and clarify the associated geometry.

The basic message is that gravity is already contained in the simplest matrix models of NC gauge theory. There is no need to invoke any new ideas. This striking mechanism takes advantage of noncommutativity in an essential way, and has no commutative analog. Furthermore, the quantization of matrix models is naturally defined by integrating over the space of matrices. We will argue that this induces the action for gravity in the spirit of [34], which suggests a natural role of UV/IR mixing. However, the vacuum equations $R_{ab} \sim 0$ are obtained even at tree level. While some freedom remains for modification of the action (in particular extra dimensions), the resulting gravity theory appears to be quite rigid. It is different from general relativity but consistent with the Newtonian limit. Moreover, some post-Newtonian corrections of general relativity appear to be reproduced, however a more detailed analysis is required. While no final judgment can be made here concerning the physics, simplicity and naturalness certainly support this mechanism.

The results of this paper should also shed new light on gravity in the IKKT model, in the presence of a noncommutative D-brane. While this model is expected to contain gravity due to its relation with string theory, an explicit identification of nontrivial geometries has proved to be difficult [13, 12]. This is discussed in section 3.

The outline of this paper is as follows. We first explain the separation of the covariant coordinates in geometric and gauge degrees of freedom, which is the essential step of our approach. This leads to a dynamical theory of Poisson manifolds, to which we associate in section 3 an effective metric. In section 3.1 we establish that this metric indeed governs the low-energy behavior of both scalar and gauge fields. The technical details for the gauge fields are lengthy and delegated to appendix A. Section 4 elaborates to some extent the physical content of the emerging gravity theory, in particular the induced Einstein-Hilbert-like action, UV/IR mixing, gravitational waves, the Newtonian limit and few examples. We conclude with discussion and outlook.

2. Gauge fields and Poisson geometry

Consider the following matrix model action for noncommutative gauge theory in 4 dimensions

$$S_{\rm YM} = -Tr[X^a, X^b][X^{a'}, X^{b'}] g_{aa'} g_{bb'}, \tag{2.1}$$

for

$$g_{aa'} = \delta_{aa'} \quad \text{or} \quad g_{aa'} = \eta_{aa'} \tag{2.2}$$

in the Euclidean resp. Minkowski case. Here the "covariant coordinates" X^a are hermitian matrices or operators acting on some Hilbert space $\mathcal{H}$. The basic symmetries of this action are the gauge symmetry

$$X^a \to U X^a U^{-1}, \qquad U \in \mathcal{U}(\mathcal{H}) \tag{2.3}$$

where $\mathcal{U}(\mathcal{H})$ are the unitary operators on $\mathcal{H}$, translational invariance $X^a \to X^a + c^a$ for $c^a \in \mathbb{R}$, and global SO(4) resp. SO(3,1) invariance. The more conventional action $Tr([X^a, X^b] - \overline{\theta}^{ab})^2$ for $\mathbb{R}^4_\theta$ [35] differs from (2.1) only by a constant shift and a topological or boundary term of the form $Tr[X^a, X^b]\,\overline{\theta}^{ab}$. We consider (2.1) to avoid introducing

JHEP12(2007)049

the constant tensor $\overline{\theta}^{ab}$ at this point, thereby stressing background-independence. This is also the type of action which is typically found in the context of string theory [7, 3]. The equations of motion are

$$[X^a, [X^{a'}, X^{b'}]]g_{aa'} = 0. \tag{2.4}$$

A particular solution is given by $X^a = \overline{Y}^a$, where the $\overline{Y}^a$ satisfy the commutation relations

$$[\overline{Y}^a, \overline{Y}^b] = \overline{\theta}^{ab}. \tag{2.5}$$

These generate the algebra $\mathcal{A} \cong \mathbb{R}^4_\theta$ of functions on the Moyal-Weyl quantum plane. Here $\overline{\theta}^{ab}$ is assumed to be constant and non-degenerate, and the $\overline{Y}^a$ have the standard Hilbert-space representations. To avoid cluttering the formulas with i we adopt the convention that θ^{ab} is purely imaginary, and similarly for the field strength F_{ab} etc. below. Another solution is given by $X^a = \overline{Y}^a \otimes 1\!\!1_n$, which will lead to $\mathfrak{u}(n)$ gauge theory.[1]

In this paper, we will focus on configurations (which need not be solutions of the e.o.m.) which are close to the "vacuum" solution $X^a = \overline{Y}^a \otimes 1\!\!1_n$. This will lead to noncommutative $\mathfrak{u}(n)$ gauge theory, or rather $\mathfrak{su}(n)$ gauge theory coupled to gravity. Hence consider small fluctuations of the form

$$X^a = \overline{Y}^a \otimes 1\!\!1_n + \mathcal{A}^a(\overline{Y}) \tag{2.6}$$

with $\mathcal{A}^a(\overline{Y}) \in \mathcal{A} \otimes M_n(\mathbb{C})$. We will replace $f(\overline{Y}) \to f(\overline{y})$ whenever $f(\overline{Y}) \in \mathcal{A}$ can be well approximated by a classical function $f(\overline{y})$. In the conventional interpretation, $\mathcal{A}^a(\overline{Y}) = \mathcal{A}^a_0(\overline{Y}) \otimes 1\!\!1_n + \mathcal{A}^a_\alpha(\overline{Y}) \otimes \tau_\alpha$ is viewed as $\mathfrak{u}(n)$-valued gauge field, where τ_α are a basis of $\mathfrak{su}(n)$. Here we will adopt a different approach, separating the trace- $\mathfrak{u}(1)$ part (i.e. the coefficient of $1\!\!1_n$) and the remaining nonabelian part as follows:

$$X^a = \; Y^a \, 1\!\!1_n \; + \; \mathcal{A}^a(Y) \; = Y^a \, 1\!\!1_n \; + \; \mathcal{A}^a_\alpha(Y) \, \tau_\alpha \; . \tag{2.7}$$

Here

$$Y^a = \overline{Y}^a + \mathcal{A}^a_0(\overline{Y}) \tag{2.8}$$

contains the full trace-$\mathfrak{u}(1)$ component, and will be interpreted as generators of a NC space $\mathcal{M}_\theta$ with general noncommutativity

$$[Y^a, Y^b] \; \equiv \; \theta^{ab}(Y) \; \approx \theta^{ab}(y) \, . \tag{2.9}$$

The other, nonabelian components $\mathcal{A}^a_\alpha(Y)$ will be considered as functions of the coordinate generators Y^a resp. y^a.

The essential point is the following: what is usually interpreted as "abelian gauge field" $\mathcal{A}^a_0$ is understood here as fluctuation of the quantum space, which determines a Poisson structure $\theta^{ab}(y)$ and eventually a metric $G^{ab}(y)$ (3.5). The remaining "nonabelian" $\mathcal{A}^a_\alpha(y) \, \tau_\alpha$ describe $\mathfrak{su}(n)$-valued gauge field. The well-known fact that the $\mathfrak{u}(1)$ and $\mathfrak{su}(n)$

[1] the rank n of therefore not determined by the matrix model but by the choice of vacuum solution. If desired, n can be controlled at least in the Euclidean case by compactifying the space and considering e.g. fuzzy $S^2 \times S^2$ or $\mathbb{C}P^2$ [26], where $\mathcal{H}$ is finite-dimensional.

JHEP12(2007)049

components cannot be completely disentangled in NC gauge theory will be understood here as coupling of the $\mathfrak{su}(n)$ gauge fields to gravity.

The physical reason why the splitting (2.7) of $\mathfrak{u}(1)$ and $\mathfrak{su}(n)$ components is appropriate will be seen by considering gauge-invariant actions such as (2.1) or (3.1). The reason is that the kinetic term in the underlying matrix-model action always involves the induced metric G^{ab} identified below. This universal coupling to a metric G^{ab} is strongly suggestive of gravity. This is based on the observation that in the framework of matrix models, all fields must be in the adjoint in order to acquire a kinetic term. However, other types of matter and low-energy gauge fields close to those required for the standard model can arise after spontaneous symmetry breaking, see e.g. [36].

Semi-classical limit: Poisson manifolds. We want to understand the geometrical significance of the various configurations (2.9). The emerging picture is that the $\mathfrak{u}(1)$ sector of the matrix model describes a dynamical theory of Poisson manifolds.

Consider generators Y^a of $\mathcal{A}$ satisfying (2.9), and assume that $\theta^{ab}(Y)$ is "close" to a smooth Poisson structure $\theta^{ab}(y)$. This defines a (local) Poisson manifold $(\mathcal{M}, \theta^{ab}(y))$ whose quantization is given by Y^a. Conversely, using a general result of Kontsevich [37] we can quantize essentially any Poisson structure[2] at least locally via such Y^a. To make this mathematically more precise, the concept of a star-product is useful. Given an isomorphism of vector spaces

$$\mathcal{C}(\mathcal{M}) \to \mathcal{A} \tag{2.10}$$

where $\mathcal{C}(\mathcal{M})$ denotes the space of functions on $\mathcal{M}$, one can define via pull-back a "star product" on $\mathcal{C}(\mathcal{M})$. Assuming that this star product has a meaningful expansion in powers of θ, the commutator of 2 elements in $\mathcal{A}$ reduces to the Poisson bracket of the classical functions on $\mathcal{M}$ to leading order in θ. More precisely, using a suitable change of variables one can choose the star product (e.g. by taking the one given in [37]) such that

$$[f, g] = i\{f, g\} \; + O(\theta^3) \; = \; \theta^{ab}(y)\, \partial_a(f)\, \partial_b(g) \; + O(\theta^3) \tag{2.11}$$

to $O(\theta^3)$. This will be important below in order to extract the semiclassical limit. In particular, this implies

$$[Y^a, f(Y)] = i\{y^a, f(y)\} \; + O(\theta^3) \; = \; \theta^{ab}(y)\, \partial_b f(y) \; + O(\theta^3)\,. \tag{2.12}$$

where y^a denotes the pull-back of Y^a.

3. Effective metric

We now show how a dynamical metric arises naturally from matrix model actions. The basic mechanism is seen most easily for scalar fields.

[2]we ignore the distinction between formal and convergent star products here.

JHEP12(2007)049

Scalar fields. In the framework of matrix models, the only possibility to obtain kinetic terms is through commutators $[X^a, \Phi] \sim \theta^{ab}(y)\partial_b\Phi + [\mathcal{A}^a, \Phi]$. Therefore only fields in the adjoint are admissible, with action

$$S[\Phi] = -Tr\, g_{aa'}[X^a, \Phi][X^{a'}, \Phi]. \tag{3.1}$$

In a configuration as in (2.7) with nontrivial background Y^a and $\mathfrak{su}(n)$-valued fluctuations

$$X^a = Y^a \otimes \mathbb{1}_n + \mathcal{A}^a(Y)\,, \tag{3.2}$$

we can obtain the commutative limit using the naive change of variables

$$\mathcal{A}^a = \theta^{ab}(y)\tilde{A}_b \tag{3.3}$$

where $\tilde{A}_a$ is antihermitian. The action then takes the form

$$\begin{aligned} S[\Phi] &\approx -Tr\, \theta^{ab}(y)\, \theta^{a'c}(y) g_{aa'}\, (\partial_b\Phi + [\tilde{A}_b, \Phi])(\partial_c\Phi + [\tilde{A}_c, \Phi]) \\ &= Tr\, G^{ab}(y)\, D_a\Phi D_b\Phi \end{aligned} \tag{3.4}$$

to leading order, defining the effective metric

$$G^{ab}(y) = -\theta^{ac}(y)\theta^{bd}(y)\, g_{cd} \tag{3.5}$$

where g_{cd} is the background metric (2.2) and $D_a = \partial_a + [\tilde{A}_a, .]$.

Some remarks are in order. We will show below that the naive substitution (3.3) is sufficient here and (3.4) is indeed the correct classical limit. An infinitesimal version of (3.5) was already obtained in [31] up to a trace contribution, which is explained in section 4.1. Furthermore, observe that

$$e^a_b(y) := -i\theta^{ac}(y)g_{cb} \tag{3.6}$$

can be interpreted as vielbein; this is consistent with the expression (3.5) for the metric G^{ab}. The antisymmetry of $\theta^{ac}(y)$ reflects the choice of a special "gauge" in comparison with the standard formulation of general relativity. Note that G^{ab} is nondegenerate if and only if the Poisson tensor $\theta^{ab}(y)$ is non-degenerate. In this paper we assume that $\theta^{ab}(y)$ is non-degenerate, even though degenerate cases are possible and are expected to be very interesting. Finally, the effective metric G^{ab} determines in particular the spectrum of the Laplacian acting on Φ; this will become important in section 4.

Nonabelian gauge fields. Now consider the commutator $[X^a, X^b]$ in the nonabelian case, for the same background. Using (2.7), we have

$$[X^a, X^b] = \theta^{ab}(Y)\, \mathbb{1}_n + \mathcal{F}^{ab}(Y) \tag{3.7}$$

where

$$\mathcal{F}^{ab} = [Y^a, \mathcal{A}^b] - [Y^b, \mathcal{A}^a] + [\mathcal{A}^a, \mathcal{A}^b] \tag{3.8}$$

is the noncommutative field strength. Our aim is to obtain the classical limit of the action (2.1), and to show that it can be interpreted as an ordinary gauge field coupled to the effective metric G^{ab}. To develop some intuition, we first give a naive, incomplete argument before embarking into the correct but less transparent Seiberg-Witten expansion.

Naive analysis. Let us try the naive[3] change of variables (3.3) which for constant θ^{ab} correctly leads to the classical limit. Using (2.12), we would obtain

$$\mathcal{F}^{ab} = \theta^{ac}(y)\theta^{bd}(y)\,\mathcal{F}_{cd} \tag{3.9}$$

where

$$\mathcal{F}_{ab} \approx \partial_a\tilde{A}_b(y) - \partial_b\tilde{A}_a(y) + [\tilde{A}_a(y),\tilde{A}_b(y)] \quad + O(\theta^{-1}\partial\theta) \; + O(\theta) \tag{3.10}$$

where $O(\theta^{-1}\partial\theta)$ stands for terms of the type $\theta^{-1}_{cd}[\theta^{db},\tilde{A}_a]$. These are small as long as

$$\theta^{-1}\,\partial\theta \ll \partial\tilde{A}_a, \tag{3.11}$$

i.e. if the variations of $\theta^{ab}(y)$ resp. G^{ab} are much slower than those of the gauge fields $\tilde{A}_c$. One can then interpret $\mathcal{F}_{ab}(y)$ as gauge field strength, which certainly holds for constant θ^{ab}. Note also that the leading term of $\mathcal{F}_{ab}$ takes values in $\mathfrak{su}(n)$, but there are $\mathfrak{u}(1)$ contributions of order θ due to e.g. $\{\mathcal{A}^a,\mathcal{A}^b\}$. Neglecting these, we would have

$$Tr(\theta^{ab}\mathcal{F}^{ab}) \approx 0$$

and the action would be

$$\begin{aligned} S_{\rm YM} &\approx -Tr\left(\theta^{ab}\,\theta^{a'b'} + \mathcal{F}^{ab}(y)\mathcal{F}^{a'b'}(y)\right) g_{aa'}g_{bb'} \\ &= Tr\left(G^{ab}(y)g_{ab} - G^{cc'}(y)\,G^{dd'}(y)\,\mathcal{F}_{cd}(y)\,\mathcal{F}_{c'd'}(y)\right) \end{aligned} \tag{3.12}$$

in the semi-classical limit. This suggests that the nonabelian gauge fields are indeed coupled as expected to the open-string metric G^{ab}. However, we need a more sophisticated analysis using the Seiberg-Witten map to establish this, because the neglected terms in (3.10) are of the same order as the coupling to the gravitational fields i.e. the connection, and the $\mathfrak{u}(1)$ terms in $\{\mathcal{A}^a,\mathcal{A}^b\}$ do in fact contribute at the leading order. This is reflected by the fact that $\mathcal{F}_{ab}$ is not gauge invariant in the commutative limit unless $\theta^{ab} = const.$

Relation with string theory. Our effective metric G^{ab} (3.5) is strongly reminiscent of the "open string metric" on noncommutative D-branes in a B-field background [3], in the Seiberg-Witten decoupling limit $\alpha' \to 0$. Our background metric g_{ab} can then be interpreted as "closed string metric" of the embedding space. However, the $\theta^{ab}(y)$ which enters our metric G^{ab} is non-constant and determined by the full $\mathfrak{u}(1)$ part of $B' = B + \mathcal{F}$ on the brane, unlike in [3]. This should be related to the symmetry $A \to A+\Lambda, B \to B - d\Lambda$ in the context of string theory as pointed out in [32, 33], where the different role of the $\mathfrak{u}(1)$ and the $\mathfrak{su}(n)$ sectors was ignored however. We will see below that G^{ab} is also the effective metric for the $\mathfrak{su}(n)$ YM-action.

3.1 Effective gauge theory and Seiberg-Witten map

In this section, we implement the separation (2.7) of the X^a in NC background Y^a and $\mathfrak{su}(n)$ gauge fields, and carefully determine the classical limit of the action (2.1). The $\mathfrak{su}(n)$-valued components of $\mathcal{A}^a$ will be expressed using a Seiberg-Witten map in terms of classical

[3] replacing this with the slightly less-naive $\mathcal{A}^a = \frac{1}{2}\{\theta^{ab}(y),\tilde{A}_a\}$ does not solve the problem

$\mathfrak{su}(n)$-valued gauge fields A_a, on a noncommutative background $\theta^{ab}(y)$ determined by the $\mathfrak{u}(1)$ components Y^a. The latter eat up the "would-be $\mathfrak{u}(1)$ gauge fields" and determine the metric $G^{ab}(y)$. Thus the full $\mathfrak{u}(1)$ sector determines the dynamical NC parameter $\theta^{ab}(y)$ and the geometry G^{ab}, while the nonabelian $\mathfrak{su}(n)$ fields are expanded to leading order in $\theta^{ab}(y)$ and couple to G^{ab}. This analysis is surprisingly involved.

Let us rewrite the nonabelian gauge fields $\mathcal{A}^a_\alpha = \mathcal{A}^a_\alpha(A_a)$ in terms of classical antihermitian $\mathfrak{su}(n)$-valued gauge fields A_a using the Seiberg-Witten map [3],[4] dropping the index α from now on. The classical gauge fields transform under $\mathfrak{su}(n)$ gauge transformations as

$$\delta_{cl} A_a = -i\partial_a\lambda + i[\lambda, A_a] = -i\partial_a\lambda + i[\lambda, \tau^\alpha]\, A_{a,\alpha}\,. \tag{3.13}$$

The appropriate SW-map for general $\theta^{ab}(y)$ is given by [38]

$$\begin{aligned}\mathcal{A}^a &= \theta^{ab}A_b - \frac{1}{2}(A_c[Y^c, \theta^{ad}A_d] + A_c F^{ca}) \quad + O(\theta^3)\\ &=: \theta^{ab}A_b + A^a_{SW,2} \quad + O(\theta^3)\end{aligned} \tag{3.14}$$

and satisfies

$$\delta_\Lambda(X^a) = i[\Lambda, Y^a + \mathcal{A}^a] = \delta_{cl}\mathcal{A}^a \tag{3.15}$$

with the NC gauge parameter

$$\Lambda = \lambda + \frac{1}{2}\theta^{ab}(\partial_a\lambda)A_b\,. \tag{3.16}$$

This means that the action (2.1) expressed in terms of A_a is invariant under the classical $\mathfrak{su}(n)$ gauge transformations acting on A_a. This in turn implies that the action can be written as a function of the ordinary $\mathfrak{su}(n)$ field strength

$$\begin{aligned}F^{ab} &:= \theta^{ac}\theta^{bd}F_{cd} = \theta^{bd}[Y^a, A_d] - \theta^{ac}[Y^b, A_c] + \theta^{ac}\theta^{bd}[A_c, A_d] \ + O(\theta^4)\\ F_{ab} &:= \partial_a A_b - \partial_b A_a + [A_a, A_b]\end{aligned} \tag{3.17}$$

In this section, we adopt the convention that indices are raised and lowered with θ^{ab} rather than a metric, e.g. $A^a := \theta^{ab}A_b$ etc. Note that it is F_{ab} rather than $\mathcal{F}^{ab}$ which has the correct classical limit as a 2-form for general $\theta^{ab}(y)$, and the classical limit can only be understood correctly in terms of F_{ab}. The reader not interested in technical details can jump to the resulting action (3.36).

Contribution to the action. We want to obtain the classical limit of the action (2.1)

$$S = -Tr(\mathcal{F}^{ab}\mathcal{F}^{ab} + 2\theta^{ab}\mathcal{F}^{ab} + \theta^{ab}\theta^{ab})$$

in terms of the A_a or F_{ab}. This requires keeping all terms of order $O(\theta^4)$. The NC field strength is

$$\begin{aligned}\mathcal{F}^{ab} &= [Y^a, \mathcal{A}^b] - [Y^b, \mathcal{A}^a] + [\mathcal{A}^a, \mathcal{A}^b]\\ &= [Y^a, A^b] - [Y^b, A^a] + [A^a, A^b] + \mathcal{F}^{ab}_{SW,2} + O(\theta^4)\end{aligned} \tag{3.18}$$

[4]The Seiberg-Witten map is used simply as a change of the field coordinates. It does not imply that we work in the framework of star-products. The non-hermitean version is used here for brevity, which is easily made hermitian.

where

$$\begin{aligned}\mathcal{F}^{ab}_{SW,2} &= [Y^a + A^a, \mathcal{A}^b_{SW,2}] + [\mathcal{A}^a_{SW,2}, Y^b + A^b] + [\mathcal{A}^a_{SW,2}\mathcal{A}^b_{SW,2}] \\ &= [X^a, \mathcal{A}^b_{SW,2}] + [\mathcal{A}^a_{SW,2}, X^b] \quad + O(\theta^4),\end{aligned} \tag{3.19}$$

since $\mathcal{A}^a_{SW,2} = O(\theta^2)$. We must carefully keep track of the $\mathfrak{u}(1)$ components of $\mathcal{F}^{ab}$ to order θ^3 and the $\mathfrak{su}(n)$ components to order θ^2. Dropping higher-order terms, one has

$$\begin{aligned}\mathcal{F}^{ab} &= [Y^a, A_d\theta^{bd}] - [Y^b, A_d\theta^{ad}] + [A^a, A^b] + \mathcal{F}^{ab}_{SW,2} \\ &= F^{ab} - A_c[Y^c, \theta^{ab}] + [A^a, A^b] - \theta^{aa'}\theta^{be'}[A_{a'}, A_{e'}] + \mathcal{F}^{ab}_{SW,2}\end{aligned} \tag{3.20}$$

using the Jacobi identity for θ^{ab}, and thus

$$\begin{aligned}S = -Tr\Big(&F^{ab}F^{ab} - 2F^{ab}A_c[Y^c, \theta^{ab}] + A_c[Y^c, \theta^{ab}][Y^d, \theta^{ab}]A_d \\ &+2(F^{ab} - A_c[Y^c, \theta^{ab}])([A^a, A^b] - \theta^{aa'}\theta^{be'}[A_{a'}, A_{e'}]) \\ &+([A^a, A^b] - \theta^{aa'}\theta^{be'}[A_{a'}, A_{e'}])^2 + 2\theta^{ab}[A^a, A^b]\Big) + S_{SW,2}\end{aligned} \tag{3.21}$$

up to $O(\theta^4)$, dropping the constant $Tr\theta^{ab}\theta^{ab}$ for now. Here

$$S_{SW,2} = -Tr\,(2\theta^{ab}\mathcal{F}^{ab}_{SW,2}) = -Tr\,(4\theta^{ab}[X^a, \mathcal{A}^b_{SW,2}])\,, \tag{3.22}$$

and we routinely drop subleading terms and use identities such as $Tr\theta^{ab}[Y^a, A^b] = 0$ since $A_a \in \mathfrak{su}(n)$. Similarly, we can set $[A, \theta] = 0$ in the $O(A^3)$ and $O(A^4)$ terms to leading order. For example,

$$[A^a, A^b] = \theta^{aa'}\theta^{bb'}[A_{a'}, A_{b'}] \quad + O(\theta^3) \tag{3.23}$$

which simplifies (3.21). Note also that the only contribution from $\theta^{ab}\mathcal{F}^{ab}$ is the NC (Poisson bracket) contribution in $\theta^{ab}[A^a, A^b]$. Therefore

$$S = -Tr\Big(F^{ab}F^{ab}-2F^{ab}A_c[Y^c, \theta^{ab}]+A_c[Y^c, \theta^{ab}][Y^d, \theta^{ab}]A_d+2\theta^{ab}[A^a, A^b]\Big)+S_{SW,2}\,. \tag{3.24}$$

After a tedious computation (see appendix A) using elementary trace-manipulations, one obtains

$$\begin{aligned}S &= -Tr\Big(F^{ab}F^{ab} - \theta^{ab}F^{ab}\theta^{cd}F_{cd} - 2\theta^{ab}F^{ad}\theta^{-1}_{dc}F^{bc} + \frac{1}{8}\theta^{ab}\theta^{ab}\theta^{cd}\theta^{ij}(F_{cd}F_{ij} + 2F_{id}F_{jc})\Big) \\ &= -Tr\Big(G^{cc'}G^{dd'}F_{cd}\,F_{c'd'} + F_{a'b'}F_{cd}(\theta^{a'a}\theta^{ab}\theta^{bb'})\theta^{cd} + 2F_{a'c}F_{b'c'}(\theta^{a'a}\theta^{ab}\theta^{bb'})\theta^{c'c} \\ &\qquad + \frac{1}{8}\theta^{ab}\theta^{ab}\theta^{cd}\theta^{ij}(F_{cd}F_{ij} + 2F_{id}F_{jc})\Big) \\ &\equiv -Tr\,G^{cc'}G^{dd'}F_{cd}\,F_{c'd'} + S_{\rm NC}\end{aligned} \tag{3.25}$$

which is exact to order $O(\theta^4)$. This action is manifestly gauge invariant, and for $\theta^{ab} = const$ it reduces to the standard YM action $S = TrF^{ab}F^{ab}$ up to boundary terms, as it should. From now on, we no longer raise or lower indices with θ^{ab}.

The "noncommutative" terms $S_{\rm NC}$ can be simplified further by considering the following dual evaluation of the 4-form resp. totally antisymmetric 4-tensor $\frac{1}{2}(F\wedge F)_{ijkl} = (F_{ij}F_{kl} - F_{il}F_{kj} - F_{lj}F_{ki})$:

$$\frac{1}{2}(F\wedge F)_{ijkl}\tilde\theta^{ij}\theta^{kl} = (F_{ij}\tilde\theta^{ij})(F_{kl}\theta^{kl}) + 2F_{il}F_{jk}\tilde\theta^{ij}\theta^{kl}. \tag{3.26}$$

We note that for $\tilde\theta^{ij} = \theta^{ik}\theta^{kl}\theta^{lj} = (\theta g\theta g\theta)^{ij}$ these are precisely the terms in $S_{\rm NC}$, and conclude

$$S_{\rm NC} = -Tr\frac{1}{2}(F\wedge F)_{ijkl}\left(\tilde\theta^{ij}\theta^{kl} + \frac{1}{8}(\theta^{ab}\theta^{ab})\theta^{ij}\theta^{kl}\right) \tag{3.27}$$

where $\theta^{ab}\theta^{ab} \equiv -G^{ab}g_{ab}$ upon reinserting g. Since $F\wedge F$ is a 4-form, it only couples to the totally antisymmetrized components $(\tilde\theta\wedge\theta)^{ijkl}$ of $\tilde\theta^{ij}\theta^{kl}$, which can be interpreted as dual 4-form. Because the space of 4-forms is one-dimensional, we must have $\tilde\theta\wedge\theta = \eta(y)\theta\wedge\theta$, and it is easy to see that (see appendix C)

$$\eta(y) = \frac{1}{4}G^{ab}g_{ab}\,. \tag{3.28}$$

Using $(F\wedge F)_{ijkl}\theta^{ij}\theta^{kl} = \frac{1}{6}(F\wedge F)_{ijkl}(\theta\wedge\theta)^{ijkl} = -\frac{1}{3}\sqrt{\det(\theta^{ab})}(F\wedge F)_{ijkl}\,\varepsilon^{ijkl}$ we finally obtain

$$S_{\rm NC} = \frac{1}{2}Tr(G^{ab}g_{ab})\sqrt{\det(\theta^{ab})}\,\frac{1}{4!}(F\wedge F)_{ijkl}\varepsilon^{ijkl}\,. \tag{3.29}$$

This reduces to a topological surface term for constant θ^{ab}, but not for general $\theta^{ab}(y)$.

Volume element. Finally we want to rewrite the trace as an integral in the semiclassical limit. According to standard Bohr-Sommerfeld quantization, the appropriate relation should be

$$(2\pi)^2 Trf(y) \sim \int\frac{1}{2}\omega^2 f(y) = \int d^4y\,\rho(y)\,f(y) \tag{3.30}$$

or equivalently

$$(2\pi)^2 Tr\sqrt{\det(\theta^{ab})} \sim \int d^4y, \tag{3.31}$$

where $\omega = i\theta^{-1}_{ab}(y)dy^a dy^b$ is the symplectic form, and $\frac{1}{2}\omega^2 = \rho(y)d^4y$ the symplectic volume element. A precise way to justify this for general (non-degenerate) $\theta^{ab}(y)$ is to require the trace property

$$\mathrm{Tr}[f,g] \sim \int\rho(y)\{f,g\} = 0 \tag{3.32}$$

up to boundary terms, which fixes $\rho(y)$ up to a constant factor. It is easy to see that $\rho(y)d^4y = \frac{1}{2}\omega^2$ indeed satisfies this requirement:

$$\begin{aligned}\int\omega^2\{f,g\} &= \int\omega^2 X_f[g] = \int\omega^2 i_{X_f}dg\\ &= -\int(i_{X_f}\omega^2)dg = 2\int(i_{X_f}\omega)\omega dg = \int df\omega dg = 0\end{aligned} \tag{3.33}$$

up to boundary terms, where X_f is the Poisson vector field generated by $\{f,.\}$. Explicitly,

$$\begin{aligned}\rho(y) &= \mathrm{Pfaff}(\mathrm{i}\theta^{-1}_{\mathrm{ab}}) = \sqrt{\det\theta^{-1}_{\mathrm{ab}}} = (\det(\mathrm{g_{ab}})\det(\mathrm{G_{ab}}))^{1/4} \\ &=: \Lambda^4_{\mathrm{NC}}(y)\end{aligned} \tag{3.34}$$

where $\Lambda_{\mathrm{NC}}(y)$ can be interpreted as "local" scale of noncommutativity.

Effective gauge action. Reinserting the constant term

$$-Tr\,\theta^{ab}\theta^{a'b'}g_{aa'}g_{bb'} = Tr\,G^{aa'}g_{aa'} = 4Tr\,\eta(y) \tag{3.35}$$

we finally obtain the classical limit of the action (2.1) in the background Y^a:

$$S_{\mathrm{YM}} = c\int d^4y\,\rho(y)tr\left(4\eta(y) - G^{cc'}G^{dd'}F_{cd}\,F_{c'd'}\right) + 2c\int \eta(y)\,trF\wedge F \tag{3.36}$$

where an overall constant c has been inserted, and $tr()$ denotes the trace over the $\mathfrak{su}(n)$ components. This is an action for a $\mathfrak{su}(n)$ gauge field coupled to a dynamical metric $G^{ab}(y)$ and the constant background metric g_{ab}.

Note that S_{YM} is invariant under local Lorentz transformations, if we consider $\eta(y)$ as a scalar function. This is remarkable, because it can be viewed as a re-summation of a Seiberg-Witten expansion in $\mathfrak{u}(1)$ from the Moyal-plane point of view, where it would appear to suffer from Lorentz violation. Therefore predictions and apparent problems for gauge theories on $\mathbb{R}^4_\theta$ due to apparent Lorentz-violation may largely disappear here.

It is fascinating to observe that $\eta(y)$ takes the place of both the cosmological "constant" and the axion, which is related to the strong CP problem. To explain that both are small are outstanding problems. The theory emerging here is expected to have important consequences on these issues, however this can only be addressed after quantum effects are taken into account. We will see that the first term in fact should *not* be interpreted as cosmological constant, rather it leads to the vacuum equations of motion at tree level (4.7), (4.19). Very similar actions have been considered from the classical point of view in [39, 40], however with an independent field replacing $\eta(y)$.

We also note that (3.34) implies the relation $(2\pi)^2\mathcal{N} = (2\pi)^2 Tr\mathbb{1} = \int d^4y\,\Lambda^4_{\mathrm{NC}}(y)$, where $\mathcal{N}$ is the dimension of the underlying Hilbert space $\mathcal{H}$ in the compact case. Therefore the local scale of noncommutativity can be interpreted as "local" dimension of $\mathcal{H}$ per coordinate volume,

$$\Lambda^4_{\mathrm{NC}} \sim \frac{(2\pi)^2\mathcal{N}}{\mathrm{Vol}}. \tag{3.37}$$

Scalar field. Similarly, we want to obtain the classical limit of the scalar action (3.1). Strictly speaking, we should also use a Seiberg-Witten map for the scalars, in order to get the correct gauge-invariant classical limit. This is given by

$$\Phi = \phi - \theta^{ab}A_a\partial_b\phi - \frac{1}{4}\theta^{ab}[A_aA_b,\phi] + O(\theta^2) \tag{3.38}$$

Noting that

$$[X^a,\Phi] = [Y^a + \theta^{ab}A_b, \phi + O(\theta)] = \theta^{ab}(\partial_b\phi + [A_b,\phi]) + O(\theta^2) \tag{3.39}$$

we obtain

$$S[\Phi] = \int d^4y\, \rho(y)\, tr\, G^{ab}(\partial_b\phi + [A_b,\phi])(\partial_b\phi + [A_b,\phi]) \tag{3.40}$$

to leading order. Therefore in the scalar case, the correct classical limit is indeed obtained by the naive analysis leading to (3.4). In particular, we obtain the same effective metric G^{ab} coupling to both scalar and gauge fields. This is of course essential for an interpretation in terms of gravity, and resolves an inconsistency for the gauge fields in [31]. Note furthermore the invariance of (3.40) under Weyl rescaling $G \to e^{\sigma}\, G$, which is usually found for the Yang-Mills sector.

The effective actions (3.36) and (3.40) almost have the standard form of gauge resp. scalar fields coupled to an external metric G^{ab}, except for the density functions $\rho(y)$ and $\eta(y)$ which depend not only on G_{ab} but also on the "background" or closed string metric g_{ab}. If we consider g_{ab} as a metric tensor, then these actions are generally covariant. However, g_{ab} is a fixed matrix in the fundamental action (2.1), where it does not make sense to transform it under a general diffeomorphisms. Thus general covariance arises only in the effective low-energy action, considering g_{ab} as a background metric which enters the Yang-Mills action only through $\det g_{ab}$ and $\eta(y)$. For fixed g_{ab}, the Yang-Mills term in (3.36) is covariant only under volume-preserving diffeomorphisms, and the "would-be topological" correction term $S_{\rm NC}$ is invariant under diffeomorphisms preserving $\eta(y)$. This is somewhat reminiscent of unimodular gravity [41], but more restrictive.

It may be tempting to recover the "missing" density factor in (3.36) by defining a slowly varying effective gauge coupling for the Yang-Mills sector,

$$\frac{1}{G^2_{\rm YM}(y)} = c\left(\frac{\det g_{ab}}{\det G_{ab}}\right)^{1/4}. \tag{3.41}$$

However this is premature and perhaps misleading at this point, because a similar $\mathfrak{su}(n)$ action will be induced at one-loop, which might have a different density factor.

4. Emergent gravity

We have shown so far that the $\mathfrak{su}(n)$ gauge fields as well as scalar fields couple (almost-) covariantly to the effective metric G^{ab}. However, we did not yet explain how the Einstein-Hilbert action or some variation thereof should arise. This appears to be difficult to achieve in the matrix-model framework, where we can write down only traces of polynomials of the covariant coordinates X^a. Moreover, adding any gauge-invariant term in the action action would also affect the $\mathfrak{su}(n)$ sector which should describe the Yang-Mills action.

We will argue that it is not necessary to add any further terms to the action, rather the gravitational action arises automatically upon quantization. The idea of induced gravity due to Sakharov [34] is crucial here; see e.g. [42] for a more recent discussion. However, the term $\int \rho(y)\eta(y)$ in (3.36) also plays an unexpected role.

Consider the quantization of the noncommutative gauge theory. The definition in terms of the matrix model actions (2.1) resp. (3.4) provides a clear quantization prescription via a (path) integral over the matrices X^a. On the other hand, we can use the description in terms of the classical actions (3.36) resp. (3.40) at least for low energies, where the classical fields are coupled covariantly to the effective metric G^{ab}. We can then use the well-known result that the one-loop effective action contains in particular the Einstein-Hilbert action.

We briefly recall this general mechanism [42]: Consider e.g. a scalar field with action $S[\Phi] = \int d^4y\,\sqrt{\tilde{g}}\,\tilde{g}^{ab}\partial_a\Phi\partial_b\Phi$ coupled to some background metric $\tilde{g}$. Upon quantization i.e. integration out ϕ up to a cutoff $\Lambda_{\rm UV}$, the leading term of the one-loop effective action is essentially given by

$$S_{1-loop} \sim \int d^4y\sqrt{\tilde{g}}\,\left(c_1\Lambda_{\rm UV}^4 + c_2\Lambda_{\rm UV}^2\,R[\tilde{g}] + O(\log(\Lambda_{\rm UV}))\right) \tag{4.1}$$

where $R[\tilde{g}]$ is the curvature scalar associated to $\tilde{g}$. It involves the Seeley-de Witt coefficients determined by the kinetic terms (see [43] §4.8). This is closely related to the spectral action principle [44], cf. [45, 46] for the Moyal-Weyl case.

Our scalar action (3.40) differs from the generally covariant form through a different power of $\det(\tilde{g})$ in the measure (3.40). This can be cast in the standard form by defining

$$\tilde{g}_{ab} = e^{\sigma}\,G_{ab}, \qquad e^{\sigma} = (\det G_{ab})^{-1/4} \tag{4.2}$$

with $\det\tilde{g} = 1$, so that

$$S[\Phi] = c\int d^4y\,(\det G_{ab})^{1/4}\,G^{ab}\partial_a\Phi\partial_b\Phi = c'\int d^4y\,\sqrt{\tilde{g}}\,\tilde{g}^{ab}\partial_a\Phi\partial_b\Phi. \tag{4.3}$$

This reflects the invariance of (3.40) under Weyl scaling. The curvature scalar of $\tilde{g}_{ab}$ is related to the one for G_{ab} by

$$R[\tilde{g}] = e^{-\sigma}\left(R[G] - 3\Delta_G\sigma - \frac{3}{2}\,G^{ab}\partial_a\sigma\partial_b\sigma\right) \tag{4.4}$$

where $e^{-\sigma} = \det(G)^{1/4}$ is somewhat reminiscent of a dilaton, and

$$\Delta_G\sigma = \nabla_G^a\partial_a\sigma = G^{ab}\partial_a\partial_b\sigma - \Gamma^c\partial_c\sigma\,. \tag{4.5}$$

Therefore (4.3) induces in particular the term

$$S_{1-loop} \sim \int d^4y\,\det(G_{ab})^{1/4}\left(R[G] - 3\Delta_G\sigma - \frac{3}{2}\,G^{ab}\partial_a\sigma\partial_b\sigma\right)\Lambda_{\rm eff}^2 \tag{4.6}$$

at one-loop. This is just an indication of what should be expected from a more detailed analysis. The $\mathfrak{su}(n)$ gauge fields will also induce at one loop terms similar to (4.6).

UV/IR mixing and gravity. It is well-known that the quantization of noncommutative field theory leads to the so-called UV/IR mixing [28 – 30]. This means in particular that the effective action contains new divergent terms with momentum dependence $\sim \frac{1}{(\theta p)^2}$, which are singular in the infrared and not contained in the bare action. This holds both for gauge fields and matter fields. Remarkably, the UV/IR mixing for gauge fields is restricted to the trace-$\mathfrak{u}(1)$ sector, at least for one loop.

Our result sheds new light on this phenomenon. We have argued using the semi-classical description that NC gauge theory induces upon quantization the Einstein-Hilbert action (4.1) for the effective metric G_{ab}, which is a function of the $\mathfrak{u}(1)$ gauge fields only, with divergent coefficients. Since these terms are not contained in the bare action, the model should *not* be naively renormalizable as a pure Yang-Mills gauge theory, and should have *new* divergences in the trace-$\mathfrak{u}(1)$ sector (and only there) at one loop. The momentum dependence of the scalar curvature R (4.18), valid for $k \ll \Lambda_{\rm NC}$, may well be responsible for the observed IR singularities in the naive $\mathfrak{u}(1)$ point of view. This shows that the essential features of the UV/IR mixing fit perfectly in our scenario and are in fact very welcome here.

It remains to be seen how much this rough picture can be substantiated. All of this underscores the importance of finite versions of NC gauge theory such as [26] which are now understood as models of Euclidean quantum gravity, and of IR-modified versions such as [47] which might suppress the gravitational sector.

Furthermore, recall that in the conventional framework, a major problem of induced gravity is that it induces huge cosmological constants. This problem is not expected to arise here, because the class of available metrics is restricted; in fact, the term $\int d^4y\, \rho(y)\eta(y)$ in (3.36) does *not* play the role of the cosmological "constant", rather it leads to the vacuum equations of motion of gravity. These are the equations of motion for the $\mathfrak{u}(1)$ degrees of freedom Y^a for $\mathcal{F}^{ab} = 0 = \Phi$, which are obtained easily from (2.4)

$$G^{ac}\partial_c\, \theta^{-1}_{ab}(y) = 0\,. \tag{4.7}$$

This will imply $R_{ab} \sim 0$ in the linearized case (4.19). Furthermore, stability of Euclidean NC spaces with similar actions as the ones considered here is rather obvious by construction [26, 48] and has been verified numerically in [49, 50], while geometrical phase transitions do occur. Moreover, flat space (2.5) remains to be a solutions even at one loop. It therefore seems quite plausible that the picture of gravity emerging from NC gauge theory may shed new light on the cosmological constant problem.

It remains to clarify the physical meaning of the metric G_{ab} and possible rescaling with e^σ, which is related to $\Lambda_{\rm NC}$ via (3.34). Furthermore, the precise form of the gravitational equations of motion should be determined. We will show that at least for small fluctuations of flat Minkowski space, the resulting gravity theory appears to be a physically acceptable modification of Einstein gravity.

Relation to previous work on Matrix models and M(atrix) theory. There is a large body of literature on Matrix-model formulations of string resp. M(atrix) theory. In particular, the IKKT for IIB string theory [7] is essentially a 10-dimensional supersymmetric version of the 4-dimensional model under consideration here, while the BFSS model [5] for M-theory includes an extra "time" dependence. The identification of gravity in these matrix models is of particular interest, and has been studied in a number of papers including [4 − 11]. What is typically considered are interactions of separated "D-objects", represented by block-matrices. A gravitational interaction is then generated at one loop, i.e. by integrating out off-diagonal blocks, reproducing leading effects of D=10 (super)gravity. However, there is also strong evidence for D=4 graviton propagators for

D=4 D-brane solutions [9, 11] of this matrix model, which is quite directly related to the present context. For other aspects see also [14 – 16]. Nevertheless, an explicit identification of the associated geometries within such matrix models and its relation with gravity has not been obtained in the literature.

The relation with our approach is as follows. In stringy language, we consider a single given NC background (a 4-brane, say), and obtain an explicit metric and effective field theory. While these brane-solutions to the matrix models are typically considered as flat (or highly symmetric), we point out that they do contain nontrivial metrics and geometry through their U(1) sector. In a higher-dimensional version, this should also shed new light on gravity in M(atrix) theory. In agreement with previous work, one-loop effects are found to be crucial to obtain the gravitational action.

4.1 Geometry, gravitational waves and $\mathfrak{u}(1)$ gauge fields

In this section we study in more detail the class of geometries available from (3.5). In particular, we consider the case of small fluctuations around a flat background $\mathbb{R}^4_\theta$ with generators $\overline{Y}^a$. This will also clarify the relation with the conventional interpretation in terms of $\mathfrak{u}(1)$ gauge fields on the canonical quantum plane $\mathbb{R}^4_\theta$.

An arbitrary $\mathfrak{u}(1)$ component of X^a in (2.7) can be written as

$$Y^a = \overline{Y}^a + \overline{\theta}^{ab} A^0_b \tag{4.8}$$

so that

$$\theta^{ab}(Y) = [Y^a, Y^b] = \overline{\theta}^{ab} + \overline{\theta}^{ac}\overline{\theta}^{bd} F^0_{cd} \tag{4.9}$$

where $F^0_{cd} = \partial_c A^0_d - \partial_d A^0_c + [A^0_c, A^0_d]$ is the abelian field strength on $\mathbb{R}^4_\theta$. Therefore the induced metric can be written in terms of the $\mathfrak{u}(1)$ gauge fields as

$$G^{ab} = -\theta^{ac} g_{cd}\theta^{bd} = -(\overline{\theta}^{ac} + \overline{\theta}^{ae}\overline{\theta}^{ch} F^0_{eh})(\overline{\theta}^{bd} + \overline{\theta}^{bf}\overline{\theta}^{dg} F^0_{fg}) g_{cd}\,. \tag{4.10}$$

Consider first the case of 2 dimensions. Then $\overline{\theta}^{ab} = \varepsilon_{ab}\overline{\theta}$ and $F^0_{ab}(y) = \varepsilon_{ab} f(y)$, therefore

$$G^{ab}_{(2D)}(y) = -g_{ab}\,\overline{\theta}^2 (1 - \overline{\theta} f(y))^2. \tag{4.11}$$

Since g_{ab} is a constant diagonal matrix, the metric is obtained automatically in isothermal coordinates, and the y-dependence of the metric is given by the y-dependence of the $\mathfrak{u}(1)$ scalar field strength. The latter is an arbitrary function off-shell. Therefore the metric $G^{ab}_{(2D)}$ describes indeed the most general metric in 2 dimensions with non-vanishing curvature, in isothermal "gauge-fixing".

In 4 dimensions, we certainly cannot obtain the most general geometry from the degrees of freedom of a $\mathfrak{u}(1)$ gauge field. However, we will show that one does obtain a class of metrics which is sufficient to describe the physical ("on-shell") degrees of freedom of gravity, more precisely gravitational waves and the Newtonian limit for an arbitrary mass distribution.

As a first check, note that gravitational waves have 2 physical degrees of freedom (helicities), as much as $\mathfrak{u}(1)$ gauge fields. We should therefore verify whether (4.10) contains indeed the 2 physical on-shell degrees of freedom of gravitational waves on Minkowski space. This was answered positively already in [31] to leading order in $\overline{\theta}^{ab}$, and is reviewed below for convenience. It strongly supports the physical viability of realizing gravity in this manner.

Gravitational waves on a flat background. Consider small fluctuations of the metric (4.10) around the metric for $\mathbb{R}^4_\theta$

$$\overline{g}^{ab} := -\overline{\theta}^{ac}\,\overline{\theta}^{bd} g_{cd}\,, \tag{4.12}$$

which is indeed flat. Keeping only the leading terms, (4.10) simplifies as

$$G^{ab} = \left(\overline{g}^{ab} + \overline{g}^{ad}\overline{\theta}^{bf}\, F^0_{df} + \overline{g}^{bd}\overline{\theta}^{af}\, F^0_{df}\right) + O(\overline{g}\overline{\theta}^2)\,. \tag{4.13}$$

This can be considered as metric fluctuations $G^{ab} = \overline{g}^{ab} - h^{ab}$ on flat Minkowski (or Euclidean) space, leading to gravitational waves determined by

$$h^{ab} = -\overline{g}^{ad}\overline{\theta}^{bf}\, F^0_{df} - \overline{g}^{bd}\overline{\theta}^{af}\, F^0_{df}\,. \tag{4.14}$$

For the inverse metric $G_{ab} = \overline{g}_{ab} + h_{ab}$ this implies

$$h_{ab} = \overline{g}_{bb'}\overline{\theta}^{b'f}\, F^0_{fa} + \overline{g}_{aa'}\overline{\theta}^{a'f}\, F^0_{fb} \tag{4.15}$$

to leading order. This is essentially the metric obtained by Rivelles [31], up to a trace contribution which arises here from the density $\rho(y)$ (3.34). Therefore the linearized picture in [31] is recovered here in a complete framework with nontrivial geometry. The linearized Ricci tensor is found to be

$$\begin{aligned} R_{ab} &= \partial^c\partial_{(b}h_{a)c} - \frac{1}{2}\partial^c\partial_c h_{ab} - \frac{1}{2}\partial_a\partial_b h \\ &= -\overline{\theta}_a{}^f\,\partial_f\partial^c F^0_{cb} - \overline{\theta}_b{}^f\,\partial_f\partial^c F^0_{ca} - \frac{1}{2}\overline{\theta}_a{}^f\,\partial^c\partial_c F^0_{bf} - \frac{1}{2}\overline{\theta}_b{}^f\,\partial^c\partial_c F^0_{af} \end{aligned} \tag{4.16}$$

where indices are raised and lowered with $\overline{g}$,

$$h = h_{ab}\overline{g}^{ab} = 2\overline{\theta}^{af}\, F^0_{fa}\,, \tag{4.17}$$

and

$$R = \overline{\theta}^{af}\,\partial^c\partial_c F^0_{af}\,. \tag{4.18}$$

This agrees (up to sign) with the results of [31], apart from the contributions from the trace part which enter in a different way. Now consider the tree-level vacuum equations of motion (4.7), which in the present context amount to $\partial^a F^0_{ab} = 0 = \partial^c\partial_c F^0_{ab}$ up to possibly corrections of order θ, i.e. the vacuum Maxwell equations for the flat metric $\overline{g}_{ab}$. As pointed out in [31], this implies that the vacuum geometries are Ricci-flat,

$$R_{ab} = 0 + O(\theta^2), \tag{4.19}$$

while the general curvature tensor R_{abcd} is first order in θ and does not vanish. This shows that the effective metric does contain the 2 physical degrees of freedom (helicities)

JHEP12(2007)049

of gravitational waves. It is quite remarkable that this is obtained at the tree level, without invoking the mechanism of induced gravity in section 4. Note that there is no cosmological constant to this order.

For completeness, we check that the Riemann tensor for plane waves is non-zero. To do this the following form of the metric fluctuations (4.15) is more convenient

$$\begin{aligned} h_{ab} &= \overline{\theta}_b{}^f\,\partial_f A_a^0 + \overline{\theta}_a{}^f\,\partial_f A_b^0 - (\partial_a A_f^0\,\overline{\theta}_b{}^f + \partial_b A_f^0\,\overline{\theta}_a{}^f) \\ &\cong \overline{\theta}_b{}^f\,\partial_f A_a^0 + \overline{\theta}_a{}^f\,\partial_f A_b^0 \end{aligned} \tag{4.20}$$

since the term in brackets has the form $\partial_a\xi_b + \partial_b\xi_a$ of an infinitesimal diffeomorphism and therefore can be dropped. Incidentally, observe that the $\mathfrak{u}(1)$ gauge transformations act as $A_a^0 \to A_a^0 + \partial_a\lambda(x)$ in the commutative limit, which leaves h_{ab} invariant; however, they do act as symplectomorphism to order θ, as discussed in section 4.3. Now consider plane-wave configurations

$$A_a^0 = E_a\, e^{ikx} \tag{4.21}$$

with

$$h_{ab} = i(\overline{\theta}_b{}^f k_f E_a + \overline{\theta}_a{}^f k_f E_b)\,. \tag{4.22}$$

Using

$$\Gamma^c_{ab} = \frac{1}{2}\overline{g}^{cd}\,(\partial_a h_{bd} + \partial_b h_{ad} - \partial_d h_{ab})\,, \tag{4.23}$$

the linearized curvature tensor is

$$R_{abc}{}^d = -i\frac{1}{2}\Big((k_c\overline{\theta}^{df} - k^d\overline{\theta}_c{}^f)k_f(k_b E_a - k_a E_b) + (k_b\overline{\theta}_a{}^f - k_a\overline{\theta}_b{}^f)k_f(k_c E^d - k^d E_c)\Big) \tag{4.24}$$

which is $O(\theta)$ and does not vanish even on-shell.

This analysis suggests in particular that gravitons should be interpreted as NC Goldstone bosons for the spontaneously broken translational invariance of $X^a \to X^a + c^a$, and gauge bosons as their nonabelian cousins.

4.2 Connection and curvature, examples

The Christoffel symbols obtained from the metric G^{ab} for general $\theta^{ab}(y)$are

$$\Gamma^c_{ab} = \frac{1}{2}G^{cd}\,(\partial_a G_{bd} + \partial_b G_{ad} - \partial_d G_{ab}) \tag{4.25}$$

which using the Jacobi identity for θ^{-1}_{ab} can be written as

$$\Gamma^c_{ab} = \frac{1}{2}\left(\theta^{cf}\partial_a\theta^{-1}_{bf} + \theta^{cf}\partial_b\theta^{-1}_{af} + G^{cd}(\theta^{-1}_{bf}g^{ff'}\partial_{f'}\theta^{-1}_{ad} + \theta^{-1}_{af}g^{ff'}\partial_{f'}\theta^{-1}_{bd})\right)\,. \tag{4.26}$$

The curvature is given as usual by

$$R_{abc}{}^d = \partial_b\Gamma^d_{ac} - \partial_a\Gamma^d_{bc} + \Gamma^e_{ac}\Gamma^d_{eb} - \Gamma^e_{bc}\Gamma^d_{ea}\,. \tag{4.27}$$

Inserting (4.26) does not provide very illuminating expressions. Note that $\theta^{ab}(y)$ is in general not covariantly constant, even though G^{ab} is.

We illustrate the nontrivial geometries emerging from NC spaces with a few examples.

JHEP12(2007)049

Manin plane. Consider the Manin plane

$$xy = qyx \tag{4.28}$$

with $|q| = 1$ and hermitian generators x, y. The underlying Poisson structure is

$$\{x, y\} = -i(q - q^{-1})\, xy =: -i\theta(x, y) \tag{4.29}$$

so that the effective metric induced by the matrix model with background metric $g_{ab} = \delta_{ab}$ resp. $g_{ab} = \eta_{ab}$ would be

$$ds^2 = -(q - q^{-1})^2\, x^2 y^2 (dx^2 \pm dy^2)\,. \tag{4.30}$$

However, keep in mind that the Manin plane might be obtained more naturally from a different matrix model with different background metric g_{ab}, with different G_{ab}.

Newtonian limit. The Newtonian limit of general relativity corresponds to static metric perturbations of the form

$$ds^2 = -c^2 dt^2 \left(1 + \frac{2U}{c^2}\right) + d\vec{x}^2 \left(1 + O\left(\frac{1}{c^2}\right)\right) \tag{4.31}$$

where $\Delta_{(3)} U = 4\pi G\rho$ and ρ is the mass density. We can indeed obtain such metrics for arbitrary static ρ, as shown in appendix B (B.14). Therefore the class of metrics G_{ab} (3.5) does contain the required degrees of freedom to describe a physically reasonable gravity theory. In fact, the degrees of freedom for G_{ab} are *precisely* those required to describe an arbitrary mass distribution. This gravity theory is therefore very economical. The Planck length is identified with $\Lambda_{\rm NC}^{-1}$ on dimensional grounds, or via (B.15) which gives $G \sim \theta$ in appropriate units.

If we us the vacuum equations of motion (4.7) which amounts to $\partial^c F_{cb} = 0$ resp. $R_{ab} = 0$ as discussed above, then (B.14) leads to

$$ds^2 = -c^2 dt^2 \left(1 + \frac{2U}{c^2}\right) + d\vec{x}^2 \left(1 - \frac{2U}{c^2}\right) \tag{4.32}$$

to leading order, as in general relativity. Therefore the leading corrections of general relativity over Newtonian gravity should be reproduced here.

Schwarzschild metric, rescaled. The Schwarzschild metric can be written in Kruskal coordinates as

$$ds^2 = r^2 \left(d\vartheta^2 + \sin^2(\vartheta) d\varphi^2\right) + \frac{4}{r}\, e^{-r} (du^2 - dv^2) \tag{4.33}$$

where $u + v = \sqrt{r-1}\, e^{(r+t)/2}$, $u - v = \sqrt{r-1}\, e^{(r-t)/2}$ and thus $u^2 - v^2 = (r-1)e^r$. This can be written as

$$G_{ab} = r^2 \tilde{G}_{ab} = r^2 \theta^{-1}_{aa'} \theta^{-1}_{bb'}\, \eta^{a'b'} \tag{4.34}$$

which almost the desired form (except for the overall scaling factor r^2) for the symplectic form

$$\theta^{-1}_{ab} dx^a \wedge dx^b = \sin(\vartheta) d\vartheta d\varphi + \frac{2}{r^{3/2}}\, e^{-r/2} du \wedge dv\,. \tag{4.35}$$

Note that the density factor $e^\sigma = (\det \tilde{G})^{1/4} = \frac{2}{r^{3/2}}\, e^{-r/2}$ is a function of r only, so that the (4.33) is indeed obtained by rescaling with a function of σ only. The (r,t) - part of the metric can easily be generalized as in (4.11). While this illustrates the nontrivial nature of metrics of the form (3.5), it turns out that this ansatz does not lead to the desired Schwarzschild-like solution, rather a different ansatz must be used; this will be described elsewhere.

4.3 Coordinates, gauge invariance and symplectomorphisms

From a semiclassical point of view, NC gauge theory provides 2 geometrical structures: 1) a Poisson structure $\theta^{ab}(x)$ and 2) a "background" (closed string) metric g_{ab}, which is used to contract the indices of the covariant coordinates. We assume here that g_{ab} is flat. There are accordingly 2 special coordinate systems:

1. Darboux coordinates where θ^{ab} is constant. Then of course the background metric $g_{ab}(x)$ is *not* given by δ_{ab} or η_{ab}, but it is still flat.

2. Cartesian coordinates w.r.t. the background metric g_{ab}. Then $\theta^{ab}(y)$ is not constant. These are the y^a coordinates used in the present paper.

Observe that G_{ab} is flat if the two coincide, thus NC gravity results in some sense from a "strain" between Darboux- and g-flat coordinates.

Now consider the gauge symmetries. The matrix-model action (2.1) is invariant under the NC gauge transformations (2.3). While their $\mathfrak{su}(n)$ components are clearly the $\mathfrak{su}(n)$ gauge transformations of the effective action (3.36), the role of the local $\mathfrak{u}(1)$ transformations is less obvious. It is well-known (see e.g. [51]) that $\mathfrak{u}(1)$ gauge transformations in the NC case act naturally as symplectomorphisms on the Poisson manifold $\mathcal{M}$, leaving $\theta^{ab}(y)$ invariant. To see this, consider the gauge transformation of a scalar function $\phi(y) \in \mathcal{A}$:

$$\phi \to \phi' = U\phi U^{-1} \tag{4.36}$$

or infinitesimally

$$\phi \to \phi' = \phi + i[\Lambda, \phi] \tag{4.37}$$

for $U = e^{i\varepsilon\Lambda}$. The semi-classical version of this action is $\phi(y) \to \phi'(y) = \phi(y) + \{\Lambda(y), \phi(y)\}$, which generates the Hamiltonian flow with generator $\Lambda(y)$ w.r.t. the Poisson structure $\theta^{ab}(y)$. Therefore $\mathfrak{u}(1)$ gauge transformations are naturally interpreted as quantization of the action of the group Symp$(\mathcal{M})$ of symplectomorphisms on $\mathcal{M}$. Due to Liouvilles theorem, Symp$(\mathcal{M})$ is a (proper) subgroup of the group of volume-preserving diffeomorphism.

Now consider the covariant coordinates X^α, which transform as

$$X^a \to X^{a\prime} = U^{-1} X^a U. \tag{4.38}$$

According to the above discussion, this can be interpreted for the $\mathfrak{u}(1)$ sector as transformation of the embedding function $X^a : \mathcal{M} \hookrightarrow \mathbb{R}^4$ under (quantized) Symp$(\mathcal{M})$. However here Symp$(\mathcal{M})$ does *not* act on any indices of e.g. nonabelian gauge fields, unlike the standard action of diffeomorphisms. Nevertheless, since the action is written in terms of classical field strength tensors with all indices properly contracted, the classical action appears to be general covariant. This is only apparent, however, since g_{ab} is a fixed background metric: The exact invariance group must preserve ρ and $\eta(y)$, which probably reduces it to Symp($\mathcal{M}$).

The role of NC gauge transformations and diffeomorphisms certainly deserves further investigations, see also [32, 52] for related discussion. It remains to be seen whether the generalized notions of symmetry developed in [17] are applicable in the context of matrix models.

5. Remarks on the quantization

The great virtue of matrix models such as (2.1) is that there is a clear concept of quantization, defined by integrating over the space of matrices. This has been extremely successful for single-matrix models, and was elaborated in the context of NC gauge theory to some extent [48]. Combined with the results of the present paper, this leads to the hope that (2.1) may provide a good definition of quantum gravity. The limit $N \to \infty$ of course remains to be a highly nontrivial issue related to renormalizability. On the other hand, the finite-dimensional matrix-models for compact "fuzzy" quantum spaces such as [26] are thus candidates for a regularized (Euclidean) gravity theory.

Furthermore, recall from section 4.2 that our model of NC gravity contains only the minimal degrees of freedom required to accomodate on-shell gravitational waves plus a mass distribution. In contrast, general relativity contains many additional off-shell and gauge degrees of freedom, leading in particular to nontrivial gauge fixing issues upon quantization. Therefore the gravity theory obtained here should be better suited for quantization.

We support this conjecture with some observations. Due to gauge invariance (2.3), the effective action after quantization should be given by similar types of matrix models, involving more complicated expressions of traces of polynomials of the X^a. Due to translational invariance, they should be expressible in terms of commutators, and therefore - in some given vacuum - the same analysis as here should establish that they can be interpreted as $\mathfrak{su}(n)$ gauge theory coupled to an effective G^{ab}, to leading order. This suggests that there should be no disastrous UV/IR mixing effect, which has been absorbed by the choice of geometric vacuum.

6. Discussion

The basic message of this paper is that gravity is an intrinsic part of the matrix-model formulation of NC gauge theory. These models describe a dynamical noncommutative space, with metric determined by the general Poisson structure. This leads to a separation of the gravity and gauge theory degrees of freedom. Quantum spaces and gravity are seen as two aspects of the same thing. Matrix models such as (2.1) thus provide a simple class of models which should be suitable for quantizing gravity along with the other fields. This clarifies the presence of gravity in string-theoretical matrix models [5, 7], however the mechanism is more general and applies in particular to 4 dimensions, as elaborated here. Also, the mechanism of spontaneous generation of fuzzy extra dimensions [36] can now be

JHEP12(2007)049

seen from the point of view of gravity. We also point out that the gravitational action will be induced upon quantization, which should explain and hopefully resolve the UV/IR mixing in NC gauge theory.

While the physical properties of the emerging gravity theory are not yet worked out, the simplicity of the mechanism is certainly striking. There remains some freedom for modification of the action, in particular via extra dimensions, but the mechanism seems to be quite rigid. In particular, the restricted class of geometries strongly suggests that the resulting gravity theory is different from general relativity, but consistent with its low-energy limit. This realizes some of the ideas in [31 − 33], with the aim to understand gravity as an emergent phenomenon of NC gauge theory in the commutative limit. It is also reminiscent to ideas in [20], in the sense that gravity is determined by noncommutativity i.e. the Poisson structure. On the other hand, this is different from other proposals [17] which aim to define a deformed (noncommutative) version of general relativity.

One may wonder how such a different interpretation of NC gauge theory is possible; after all, there seems to be nothing wrong with the "old" gauge theory point of view. From that perspective, what we have done is to perform a Seiberg-Witten map from constant $\overline{\theta}^{ab}$ to a general $\theta^{ab}(y)$, to leading order in $\theta^{ab}(y)$ but exact in $\delta\theta^{ab} = \theta^{ab}(y) - \overline{\theta}^{ab}(y)$. This "eats up" the $\mathfrak{u}(1)$ gauge fields and moves them into the metric $G^{ab}(y)$. In the conventional gauge theory point of view, $\delta\theta^{ab}$ is the $\mathfrak{u}(1)$ field strength, which decouples from the $\mathfrak{su}(n)$ gauge degrees of freedom to leading order but cannot be disentangled exactly. We determined the precise coupling between these $\mathfrak{u}(1)$ and $\mathfrak{su}(n)$ degrees of freedom, and showed that it should be interpreted as gravitational coupling. This casts the basic observations of [31] in a complete framework, generalized to notrivial geometries and nonabelian gauge fields. The basic idea of gravity emerging form NC gauge theory was also put forward in [32, 33], in a somewhat different approach without identifying the metric (3.5).

There are many further directions to explore. First, the main results of this paper also apply to dimension different from 4, and should generalize in particular to the case of NC "submanifolds" embedded in higher dimensions. Then the closed string metric g_{ab} is the induced metric on the submanifold, and no longer flat in general. Therefore the class of effective metrics obtained in this case may be larger. Notice also that extra dimensions can be viewed as additional (possibly interacting) scalars as in (3.1); a particularly interesting example would be the matrix model for $N = 4$ NCSYM considered e.g. in [9, 11]. Other types of matrix model actions should also be explored, such as DBI-like actions. Fermions should of course be included in these models, which will be studied elsewhere. This will also allow to study the relation with the framework of the spectral action [44]. The quantization and loop effects should be worked out. Finally, it is of course essential to explore the physical viability of this NC gravity.

Acknowledgments

This paper is dedicated to the memory of Julius Wess, who took a leading role in the development of this circle of ideas in physics, and provided essential support and an open spirit.

I would like to thank in particular H. Grosse for discussions on various aspects of this work, as well as C-S. Chu, J. Wess, and H.-S. Yang for stimulating exchange on various aspects and related ideas. I am also grateful to M. Wohlgenannt for pointing out [31] to me. This work was supported by the FWF Project P18657.

A. Derivation of the effective action to leading order

To shorten the notation we only consider the Euclidean case $g_{ab} = \delta_{ab}$ here, and adopt a notation where repeated indices are summed irrespective of their position; for example, $\theta^{ab}\theta^{ab} \equiv \sum_{a,b} \theta^{ab}\theta^{ab}$. The Minkowski case is obtained by obvious replacements.

Furthermore, we adopt the convention in this appendix to rise and lower indices with θ^{ab} resp. θ^{-1}_{ab} rather than the metric, e.g. $A^a = \theta^{ab} A_b$.

Useful identities

The "commutative" field strength is defined by

$$\begin{aligned} F^{ab} &= \theta^{ac}\theta^{bd}F_{cd} = \theta^{ac}\theta^{bd}(\partial_c A_d - \partial_d A_c) + \theta^{ac}\theta^{bd}[A_c, A_d] \\ &= \theta^{bd}[Y^a, A_d] - \theta^{ac}[Y^b, A_c] + \theta^{ac}\theta^{bd}[A_c, A_d] \end{aligned} \tag{A.1}$$

while we define the "noncommutative" field strength as

$$\begin{aligned} \mathcal{F}^{ab} &= [X^a, X^b] - \theta^{ab} = [Y^a, \mathcal{A}^b] - [Y^b, \mathcal{A}^a] + [\mathcal{A}^a, \mathcal{A}^b] \\ &= [X^a, \mathcal{A}^b] - [X^b, \mathcal{A}^a] - [\mathcal{A}^a, \mathcal{A}^b]\,. \end{aligned} \tag{A.2}$$

The leading terms are

$$\begin{aligned} \mathcal{F}^{ab} &= [Y^a, A_d\theta^{bd}] - [Y^b, A_d\theta^{ad}] + [A^a, A^b] \\ &= F^{ab} + ([Y^a, \theta^{bd}] - [Y^b, \theta^{ad}])A_d + [A_d\theta^{ad}, A_e\theta^{be}] - \theta^{ad}\theta^{be}[A_d, A_e] \\ &= F^{ab} - A_d[Y^d, \theta^{ab}] + [A^a, A^b] - \theta^{aa'}\theta^{be'}[A_{a'}, A_{e'}] \end{aligned} \tag{A.3}$$

up to corrections of order $O(\theta^3)$, hence omitting $\mathcal{F}^{ab}_{SW,2}$ here.

A useful identity is

$$2\theta^{ab}[Y^a, [Y^b, X]] = \theta^{ab}([Y^a, [Y^b, X]] - [Y^b, [Y^a, X]]) = \theta^{ab}[\theta^{ab}, X]\,. \tag{A.4}$$

A similar identity is the following:

$$\begin{aligned} \theta^{ab}[Y^a, \theta^{cb}] &= -\theta^{ab}[Y^c, \theta^{ba}] - \theta^{ab}[Y^b, \theta^{ac}] \\ &= -\theta^{ab}[Y^c, \theta^{ba}] + \theta^{ab}[Y^a, \theta^{bc}] \end{aligned}$$

therefore

$$\theta^{ab}[Y^a, \theta^{cb}] = \frac{1}{2}\theta^{ab}[Y^c, \theta^{ab}]\,. \tag{A.5}$$

In particular,

$$\theta^{ab}A_c A_d[Y^d, [Y^a, \theta^{cb}]] = \frac{1}{2}\theta^{ab}A_c A_d[Y^d, [Y^c, \theta^{ab}]]\,. \tag{A.6}$$

Bianci identity and applications. The noncommutative Bianci identity for $\mathcal{F}$ is obtained from

$$[X^a,\mathcal{F}^{bc}]+[X^b,\mathcal{F}^{ca}]+[X^c,\mathcal{F}^{ab}] = -[X^a,\theta^{bc}]-[X^b,\theta^{ca}]-[X^c,\theta^{ab}]$$
$$= -[\mathcal{A}^a,\theta^{bc}]-[\mathcal{A}^b,\theta^{ca}]-[\mathcal{A}^c,\theta^{ab}]\,. \qquad \text{(A.7)}$$

Together with the antisymmetry of θ^{ab}, it follows that

$$\theta^{ab}[X^a,\mathcal{F}^{cb}] = \theta^{ab}\left(-[X^c,\mathcal{F}^{ba}]-[X^b,\mathcal{F}^{ac}]-[\mathcal{A}^a,\theta^{cb}]-[\mathcal{A}^b,\theta^{ac}]-[\mathcal{A}^c,\theta^{ba}]\right)$$
$$= \theta^{ab}\left(-[X^c,\mathcal{F}^{ba}]-[X^a,\mathcal{F}^{cb}]-[\mathcal{A}^a,\theta^{cb}]-[\mathcal{A}^a,\theta^{cb}]+[\mathcal{A}^c,\theta^{ab}]\right) \qquad \text{(A.8)}$$

which implies

$$\theta^{ab}[X^a,\mathcal{F}^{cb}] = \frac{1}{2}\theta^{ab}([X^c,\mathcal{F}^{ab}]+[A^c,\theta^{ab}])-\theta^{ab}[A^a,\theta^{cb}]\,. \qquad \text{(A.9)}$$

Using $[Y^c,A^b]+[A^c,A^b]=\mathcal{F}^{cb}+[Y^b,A^c]$ this gives

$$\begin{aligned}
\theta^{ab}[X^a,[Y^c,A^b]+[A^c,A^b]]] &= \theta^{ab}[X^a,\mathcal{F}^{cb}]+\theta^{ab}[X^a,[Y^b,A^c]] \\
&= \frac{1}{2}\theta^{ab}([X^c,\mathcal{F}^{ab}]+[A^c,\theta^{ab}])+\theta^{ab}[Y^a,[Y^b,A^c]] \\
&\quad +\theta^{ab}[A^a,[Y^b,A^c]]-\theta^{ab}[A^a,\theta^{cb}] \\
&= \frac{1}{2}\theta^{ab}([X^c,\mathcal{F}^{ab}]+[A^c,\theta^{ab}])+\frac{1}{2}\theta^{ab}[\theta^{ab},A^c] \\
&\quad +\theta^{ab}[A^a,[Y^b,A^c]]-\theta^{ab}[A^a,\theta^{cb}] \\
&= \theta^{ab}\left(\frac{1}{2}[X^c,\mathcal{F}^{ab}]+[Y^a,[A^c,A^b]]-[A^c,[A^a,Y^b]]-[A^a,\theta^{cb}]\right)
\end{aligned}$$

so that

$$\theta^{ab}[X^a,[Y^c,A^b]] = \frac{1}{2}\theta^{ab}[X^c,\mathcal{F}^{ab}]-\theta^{ab}[A^c,[A^a,Y^b]]-\theta^{ab}[A^a,[A^c,A^b]]-\theta^{ab}[A^a,\theta^{cb}]$$

which using

$$\theta^{ab}[A^a,[A^c,A^b]] = -\theta^{ab}[A^c,[A^b,A^a]]+\theta^{ab}[A^a,[A^b,A^c]] \qquad \text{(A.10)}$$

thus

$$\theta^{ab}[A^a,[A^c,A^b]] = \frac{1}{2}\theta^{ab}[A^c,[A^a,A^b]] \qquad \text{(A.11)}$$

gives

$$\begin{aligned}
\theta^{ab}[X^a,[Y^c,A^b]] &= \frac{1}{2}\theta^{ab}[X^c,\mathcal{F}^{ab}]-\theta^{ab}[A^c,[A^a,Y^b]]-\frac{1}{2}\theta^{ab}[A^c,[A^a,A^b]]-\theta^{ab}[A^a,\theta^{cb}] \\
&= \frac{1}{2}\theta^{ab}[X^c,\mathcal{F}^{ab}]-\frac{1}{2}\theta^{ab}[A^c,F^{ab}]-\theta^{ab}[A^a,\theta^{cb}]\,.
\end{aligned} \qquad \text{(A.12)}$$

Other useful relations. Here we collect some identities which hold up to some required order or θ.

Let us introduce the notation

$$[Y_a, f] := \theta^{-1}_{ab}\,[Y^b, f] = \partial_a f \quad + O(\theta) \tag{A.13}$$

which allows to write

$$[Y^a, f][Y_a, g] = \theta^{ab}\partial_b f \partial_a g = -i\{f, g\} \quad + O(\theta^2) \tag{A.14}$$

to leading order, which in the abelian case coincides with $-[f, g] + O(\theta^2)$. This gives

$$\begin{aligned}
\theta^{ab}[X^a, A_c](F^{cb}+[Y^c, A^b]) &= \theta^{ab}(F^a{}_c + [Y_c, A_e]\theta^{ae})(F^{cb} + [Y^c, A^b]) \\
&= \theta^{ab}(F^a{}_c F^{cb}+[Y_c, A_e]\theta^{ae}[Y^c, A^b]+F^a{}_c[Y^c, A^b] + F^{cb}[Y_c, A_e]\theta^{ae}) \\
&=\theta^{ab}(F^a{}_c F^{cb} + \theta^{ae} i\{A_e, A^b\} - F^{cb}A_e[Y_c, \theta^{ae}]) \\
&= \theta^{ab}F^{ad}\theta^{-1}_{dc}F^{bc} + \theta^{ab}F^{ad}\theta^{-1}_{dc}A_e[Y^c, \theta^{eb}] + \theta^{ab}\theta^{ae} i\{A_e, A^b\}
\end{aligned} \tag{A.15}$$

up to $O(\theta^4)$. Similarly, one finds

$$\begin{aligned}
\theta^{ab}(A_d[X^a, A_c][Y^c, \theta^{db}] + F^{ad}\theta^{-1}_{dc}A_e[Y^c, \theta^{eb}]) &= \theta^{ab}A_d([X^a, A_c] - F^a{}_c)[Y^c, \theta^{db}]) \\
&= \theta^{ab}A_d[Y_c, A_e]\theta^{ae}[Y^c, \theta^{db}] \\
&\sim \theta^{ab}\theta^{ae}A_d[A_e, \theta^{db}]\,.
\end{aligned} \tag{A.16}$$

To evaluate the contributions cubic in A, we will need

$$\begin{aligned}
Tr\theta^{ab}\theta^{ab}A_d[\mathcal{F}^{cd}, A_c] &= -Tr\theta^{ab}\theta^{ab}[A_d, A_c]\mathcal{F}^{cd} \\
&= -Tr\theta^{ab}\theta^{ab}[A_d, A_c]([Y^c, A^d] - [Y^d, A^c] + [A^c, A^d]) \\
&= -Tr\theta^{ab}\theta^{ab}[A_d, A_c](2[Y^c, A^d] + [A^c, A^d])\,.
\end{aligned} \tag{A.17}$$

The first term gives

$$\begin{aligned}
Tr\theta^{ab}\theta^{ab}[A_d, A_c][Y^c, A^d] &= Tr - \theta^{ab}\theta^{ab}A_d[[Y^c, A^d], A_c] \\
&= Tr\theta^{ab}\theta^{ab}A_d([[A^d, A_c], Y^c] + [[A_c, Y^c], A^d]) \\
&= Tr\theta^{ab}\theta^{ab}(-[A_d, Y^c][A^d, A_c] - [A_d, A^d][A_c, Y^c])\,.
\end{aligned} \tag{A.18}$$

To proceed, consider

$$\begin{aligned}
Tr\theta^{ab}\theta^{ab}[A^d, Y^c][A_d, A_c] &= Tr\theta^{ab}\theta^{ab}[\theta^{de}A_e, Y^c][A_d, A_c] \\
&= Tr\theta^{ab}\theta^{ab}([\theta^{de}, Y^c]A_e + [A_e, Y^c]\theta^{de})[A_d, A_c] \\
&= Tr\theta^{ab}\theta^{ab}([\theta^{de}, Y^c]A_e[A_d, A_c] - [A_e, Y^c][A^e, A_c]) \\
&= -Tr\theta^{ab}\theta^{ab}[A_e, Y^c][A^e, A_c]
\end{aligned} \tag{A.19}$$

dropping terms of order $O(\theta^5)$ and using (A.21) below, which can be obtained by considering

$$\begin{aligned}
Tr\theta^{ab}\theta^{ab}[A_c, A_d]A_e[Y^e, \theta^{cd}] &= Tr\theta^{ab}\theta^{ab}[A_c, A_d]A_e(-[Y^c, \theta^{de}] + [Y^d, \theta^{ce}]) \\
&= -2Tr\theta^{ab}\theta^{ab}[A_c, A_d]A_e[Y^c, \theta^{de}] \\
&= -2Tr\theta^{ab}\theta^{ab}[A_d, A_e]A_c[Y^c, \theta^{de}],
\end{aligned} \tag{A.20}$$

routinely dropping terms of the type $Tr\theta^4 f(x)[A_a, A_b]_{n.a.}$ under the trace. This implies that

$$Tr\theta^{ab}\theta^{ab}[A_c, A_d]A_e[Y^e, \theta^{cd}] = Tr\theta^{ab}\theta^{ab}[A_c, A_d]A_e[Y^c, \theta^{de}] = 0\,. \tag{A.21}$$

Therefore (A.18) gives

$$Tr\theta^{ab}\theta^{ab}[A_d, A_c][Y^c, A^d] = Tr\theta^{ab}\theta^{ab}(-[Y^c, A^d][A_d, A_c] - [A_d, A^d][A_c, Y^c]) \tag{A.22}$$

which implies

$$Tr\theta^{ab}\theta^{ab}[A_d, A_c][Y^c, A^d] = -\frac{1}{2}Tr\theta^{ab}\theta^{ab}[A^d, A_d][Y^c, A_c]\,. \tag{A.23}$$

Similarly,

$$\begin{aligned}
-Tr\theta^{ab}\theta^{ab}[A_d, A_c][A^c, A^d] &= Tr\theta^{ab}\theta^{ab}A_c[A_d, [A^c, A^d]] \\
&= Tr\theta^{ab}\theta^{ab}A_c(-[A^c, [A^d, A_d]] - [A^d, [A_d, A^c]]) \\
&= Tr\theta^{ab}\theta^{ab}([A^c, A_c][A^d, A_d] + [A^d, A_c][A_d, A^c]) \\
&= Tr\theta^{ab}\theta^{ab}([A^c, A_c][A^d, A_d] - [A_d, A_c][A^d, A^c])
\end{aligned} \tag{A.24}$$

implies

$$Tr\theta^{ab}\theta^{ab}[A_d, A_c][A^c, A^d] = -\frac{1}{2}Tr\theta^{ab}\theta^{ab}[A^c, A_c][A^d, A_d]\,. \tag{A.25}$$

Putting this together, (A.17) can be written as

$$Tr\theta^{ab}\theta^{ab}A_d[\mathcal{F}^{cd}, A_c] = \frac{1}{2}Tr\theta^{ab}\theta^{ab}(2[A^d, A_d][Y^c, A_c] + [A^c, A_c][A^d, A_d])\,. \tag{A.26}$$

Evaluation of the contributions

second-order Seiberg-Witten contribution

Let us write the second-order Seiberg-Witten contributions (3.22):

$$\begin{aligned}
S_{SW,2} &= 2Tr\theta^{ab}[X^a, A_c([Y^c, A^b] + F^{cb})] \\
&= 2Tr\theta^{ab}\Big([X^a, A_c](F^{cb} + [Y^c, A^b]) + A_c[X^a, (\mathcal{F}^{cb} + [Y^c, A^b] + A_d[X^d, \theta^{cb}])]\Big) \\
&= 2Tr\theta^{ab}\Big([X^a, A_c](F^{cb} + [Y^c, A^b]) + A_c[X^a, (\mathcal{F}^{cb} + [Y^c, A^b] + A_d[X^d, \theta^{cb}])]\Big)
\end{aligned}$$

where we used (A.3)

$$\mathcal{F}^{ab} = F^{ab} - A_d[Y^d, \theta^{ab}] \ + O(\theta^3), \tag{A.27}$$

noting (3.23). The second line can be simplified using (A.12)

$$\theta^{ab}[X^a, [Y^c, A^b]] = \frac{1}{2}\theta^{ab}[X^c, \mathcal{F}^{ab}] - \frac{1}{2}\theta^{ab}[A^c, F^{ab}] - \theta^{ab}[A^a, \theta^{cb}] \tag{A.28}$$

and (A.9)

$$\theta^{ab}[X^a, \mathcal{F}^{cb}] = \frac{1}{2}\theta^{ab}([X^c, \mathcal{F}^{ab}] + [A^c, \theta^{ab}]) - \theta^{ab}[A^a, \theta^{cb}] \tag{A.29}$$

so that

$$
\begin{aligned}
S_{SW,2} = Tr\theta^{ab}\Big(& 2[X^a, A_c](F^{cb} + [Y^c, A^b]) \\
& +2A_c[X^c, \mathcal{F}^{ab}] - A_c\theta^{ab}[A^c, F^{ab}] + A_c[A^c, \theta^{ab}] - 4A_c[A^a, \theta^{cb}] \\
& +2A_c[X^a, A_d][Y^d, \theta^{cb}] + 2A_cA_d[X^a, [X^d, \theta^{cb}]]\Big) .
\end{aligned} \tag{A.30}
$$

Now

$$
\begin{aligned}
\theta^{ab}A_cA_d[Y^a, [Y^d, \theta^{cb}]] &= \theta^{ab}A_cA_d[Y^d, [Y^a, \theta^{cb}]] + \theta^{ab}A_cA_d[\theta^{ad}, \theta^{cb}] \\
&= \frac{1}{2}Tr\theta^{ab}A_cA_d[Y^d, [Y^c, \theta^{ab}]] + \theta^{ab}A_cA_d[\theta^{ad}, \theta^{cb}]
\end{aligned} \tag{A.31}
$$

using (A.6), which implies

$$
\theta^{ab}A_cA_d[X^a, [X^d, \theta^{cb}]] = \frac{1}{2}Tr\theta^{ab}A_cA_d[X^d, [X^c, \theta^{ab}]] + \theta^{ab}A_cA_d[\theta^{ad}, \theta^{cb}] \ + O(\theta^5) \tag{A.32}
$$

hence

$$
\begin{aligned}
S_{SW,2} = Tr\theta^{ab}\Big(& 2[Y^a, A_c](F^{cb} + [Y^c, A^b]) \\
& +2A_c[X^c, \mathcal{F}^{ab}] - A_c\theta^{ab}[A^c, F^{ab}] + A_c[A^c, \theta^{ab}] - 4A_c[A^a, \theta^{cb}] \\
& +2A_d[X^a, A_c][Y^c, \theta^{db}] + A_cA_d[X^d, [X^c, \theta^{ab}]] + 2A_cA_d[\theta^{ad}, \theta^{cb}]\Big) .
\end{aligned} \tag{A.33}
$$

The first line can be written using (A.15) which gives

$$
\begin{aligned}
S_{SW,2} = Tr\theta^{ab}\Big(& 2F^{ad}\theta^{-1}_{dc}F^{bc} + 2F^{ad}\theta^{-1}_{dc}A_e[Y^c, \theta^{eb}] + 2\theta^{ae}i\{A_e, A^b\} \\
& +2A_c[X^c, \mathcal{F}^{ab}] - A_c\theta^{ab}[A^c, F^{ab}] + A_c[A^c, \theta^{ab}] - 4A_c[A^a, \theta^{cb}] \\
& +2A_d[X^a, A_c][Y^c, \theta^{db}] + A_cA_d[X^d, [X^c, \theta^{ab}]] + 2A_cA_d[\theta^{ad}, \theta^{cb}]\Big) .
\end{aligned} \tag{A.34}
$$

Now using (A.16) this becomes

$$
\begin{aligned}
S_{SW,2} = Tr\theta^{ab}\Big(& 2F^{ad}\theta^{-1}_{dc}F^{bc} + 2\theta^{ae}i\{A_e, A^b\} + 2\theta^{ae}A_d[A_e, \theta^{db}] \\
& +2A_c[X^c, \mathcal{F}^{ab}] - A_c\theta^{ab}[A^c, F^{ab}] + A_c[A^c, \theta^{ab}] - 4A_c[A^a, \theta^{cb}] \\
& +A_cA_d[X^d, [X^c, \theta^{ab}]] + 2A_cA_d[\theta^{ad}, \theta^{cb}]\Big) .
\end{aligned}
$$

Replacing $Tr\theta^{ab}\theta^{ae}i\{A_e, A^b\} \rightarrow Tr\theta^{ab}\theta^{ae}[A_e, A^b]$ and noting

$$
\begin{aligned}
& \theta^{ab}\left(\theta^{ae}[A_e, A^b] + \theta^{ae}A_d[A_e, \theta^{db}] - 2A_c[A^a, \theta^{cb}] + A_cA_d[\theta^{ad}, \theta^{cb}]\right) \\
&= \theta^{ab}\left(\theta^{ad}[A_d, A^b] + \theta^{ad}A_c[A_d, \theta^{cb}] - 2A_c[A^a, \theta^{cb}] + A_c[A_d\theta^{ad}, \theta^{cb}] - \theta^{ad}A_c[A_d, \theta^{cb}]\right) \\
&= \theta^{ab}\theta^{ad}[A_d, A^b] - \theta^{ab}A_c[A^a, \theta^{cb}] \\
&= \theta^{ab}\theta^{ad}[A_d, A^b] - \theta^{ab}A_d[\theta^{da}, A^b] \\
&= \theta^{ab}[A^a, A^b]
\end{aligned} \tag{A.35}
$$

(note: only the abelian component involving the Poisson bracket contributes) and

$$Tr\left(A_c\theta^{ab}[A^c, F^{ab}]\right) = -Tr\left([A^c, A_c]\theta^{ab}F^{ab}\right) \tag{A.36}$$

(since only the nonabelian terms survive), we obtain

$$\begin{aligned} S_{SW,2} = Tr\theta^{ab}\Big(&2F^{ad}\theta^{-1}_{dc}F^{bc} + 2[A^a, A^b] + A_c[A^c, \theta^{ab}] \\ &+2A_c[X^c, \mathcal{F}^{ab}] + [A^c, A_c]\theta^{ab}F^{ab} + A_cA_d[X^d, [X^c, \theta^{ab}]]\Big)\,. \end{aligned} \tag{A.37}$$

Now we use

$$\begin{aligned} A_c[X^c, \mathcal{F}^{ab}] &= A_c[X^c, F^{ab} - A_d[Y^d, \theta^{ab}]] \\ &= A_c[X^c, F^{ab}] - A_c[X^c, A_d[Y^d, \theta^{ab}]] \\ &= A_c[X^c, F^{ab}] - A_c[X^c, A_d[X^d, \theta^{ab}]] \end{aligned} \tag{A.38}$$

(to $O(\theta^4)$) using (A.3), and obtain

$$\begin{aligned} S_{SW,2} = Tr\theta^{ab}\Big(&2F^{ad}\theta^{-1}_{dc}F^{bc} + 2\theta^{ab}[A^a, A^b] + 2A_c[X^c, F^{ab}] + [A^c, A_c]\theta^{ab}F^{ab} + A_c[A^c, \theta^{ab}] \\ &+A_cA_d[X^d, [X^c, \theta^{ab}]] - 2A_c[X^c, A_d[X^d, \theta^{ab}]]\Big). \end{aligned} \tag{A.39}$$

Using partial integration, we have

$$Tr\theta^{ab}A_d[X^d, A_c[X^c, \theta^{ab}]] = -TrA_d[X^d, \theta^{ab}]A_c[X^c, \theta^{ab}] - Tr\theta^{ab}[X^d, A_d]A_c[X^c, \theta^{ab}] \tag{A.40}$$

and

$$\begin{aligned} Tr\theta^{ab}A_cA_d[X^d, [X^c, \theta^{ab}]] &= Tr\theta^{ab}A_d[X^d, A_c[X^c, \theta^{ab}]] - Tr\theta^{ab}A_d[X^d, A_c][X^c, \theta^{ab}] \\ &= Tr - \theta^{ab}[X^d, A_d]A_c[X^c, \theta^{ab}] - [X^d, \theta^{ab}]A_dA_c[X^c, \theta^{ab}] \\ &\quad -Tr\theta^{ab}A_d[X^d, A_c][X^c, \theta^{ab}] \end{aligned}$$

therefore

$$\begin{aligned} &Tr - 2\theta^{ab}A_d[X^d, A_c[X^c, \theta^{ab}]] + \theta^{ab}A_cA_d[X^d, [X^c, \theta^{ab}]] \\ &\quad = TrA_d[X^d, \theta^{ab}]A_c[X^c, \theta^{ab}] + \theta^{ab}[X^d, A_d]A_c[X^c, \theta^{ab}] - \theta^{ab}A_d[X^d, A_c][X^c, \theta^{ab}]\,. \end{aligned}$$

Consider the term

$$\begin{aligned} -2Tr\theta^{ab}A_d[X^d, A_c][X^c, \theta^{ab}] &= Tr\theta^{ab}[X^c, A_d][X^d, A_c]\theta^{ab} + \theta^{ab}\theta^{ab}A_d[X^c, [X^d, A_c]] \\ &= Tr\theta^{ab}[X^c, A_d][X^d, A_c]\theta^{ab} + \theta^{ab}\theta^{ab}A_d[X^d, [X^c, A_c]] \\ &\quad +\theta^{ab}\theta^{ab}A_d[(\theta^{cd} + \mathcal{F}^{cd}), A_c] \\ &= Tr\theta^{ab}[X^c, A_d][X^d, A_c]\theta^{ab} - \theta^{ab}\theta^{ab}[X^d, A_d][X^c, A_c] \\ &\quad -2\theta^{ab}A_d[X^d, \theta^{ab}][X^c, A_c] + \theta^{ab}\theta^{ab}A_d[(\theta^{cd} + \mathcal{F}^{cd}), A_c] \end{aligned}$$

(using partial integration again), which gives

$$\begin{aligned} &-Tr\theta^{ab}A_d[X^d, A_c][X^c, \theta^{ab}] + \theta^{ab}A_d[X^d, \theta^{ab}][X^c, A_c] \\ &= Tr\frac{1}{2}\theta^{ab}[X^c, A_d][X^d, A_c]\theta^{ab} - \frac{1}{2}\theta^{ab}\theta^{ab}[X^d, A_d][X^c, A_c] + \frac{1}{2}\theta^{ab}\theta^{ab}A_d[(\theta^{cd} + \mathcal{F}^{cd}), A_c] \end{aligned}$$

and we obtain

$$
\begin{aligned}
& Tr - 2\theta^{ab} A_d[X^d, A_c[X^c, \theta^{ab}]] + \theta^{ab} A_c A_d[X^d, [X^c, \theta^{ab}]] \\
& \quad = Tr A_d[X^d, \theta^{ab}] A_c[X^c, \theta^{ab}] + \frac{1}{2}\theta^{ab}[X^c, A_d][X^d, A_c]\theta^{ab} - \frac{1}{2}\theta^{ab}\theta^{ab}[X^d, A_d][X^c, A_c] \\
& \qquad + \frac{1}{2}\theta^{ab}\theta^{ab} A_d[(\theta^{cd} + \mathcal{F}^{cd}), A_c]\,. \qquad \text{(A.41)}
\end{aligned}
$$

Inserting this into (A.39) and using

$$
\begin{aligned}
& Tr\left(\theta^{ab} A_c[A^c, \theta^{ab}] - \frac{1}{2}\theta^{ab}\theta^{ab} A_c[\theta^{cd}, A_d]\right) \\
& \quad = Tr\left(\theta^{ab} A_d[A^d, \theta^{ab}] + \frac{1}{2}\theta^{ab}\theta^{ab}[A^d, A_d] + \frac{1}{2}\theta^{ab}\theta^{ab}\theta^{cd}[A_c, A_d]\right) \\
& \quad = Tr\left(\frac{1}{2}\theta^{ab}\theta^{ab}\theta^{cd}[A_c, A_d]\right) \qquad \text{(A.42)}
\end{aligned}
$$

(again only the abelian contribution from the Poisson-bracket survives) gives

$$
\begin{aligned}
S_{SW,2} = Tr\Big(& 2\theta^{ab} F^{ad} \theta^{-1}_{dc} F^{bc} + 2\theta^{ab}[A^a, A^b] + 2\theta^{ab} A_c[X^c, F^{ab}] + [A^c, A_c]\theta^{ab} F^{ab} \\
& + \frac{1}{2}\theta^{ab}\theta^{ab} A_d[\mathcal{F}^{cd}, A_c] + A_d[X^d, \theta^{ab}] A_c[X^c, \theta^{ab}] \\
& + \frac{1}{2}\theta^{ab}\theta^{ab}([X^d, A_c][X^c, A_d] - [X^d, A_d][X^c, A_c] + \theta^{cd}[A_c, A_d])\Big)\,. \qquad \text{(A.43)}
\end{aligned}
$$

Now observe that

$$
[Y^c, A_d][Y^d, A_c] - \theta^{cd}[A_d, A_c] = \theta^{ci}\theta^{dj}\partial_i A_d \partial_j A_c - \theta^{cd}\theta^{ij}\partial_i A_d \partial_j A_c - \theta^{cd}[A_d, A_c]_{n.a.} \qquad \text{(A.44)}
$$

where $[A_d, A_c]_{n.a.}$ stands for commutator of the nonabelian components. We can drop terms of the type $Tr\theta^4 f(x)[A_a, A_b]_{n.a.}$ under the trace. Therefore

$$
\begin{aligned}
[X^c, A_d][X^d, A_c] - \theta^{cd}[A_d, A_c] &= \theta^{ci}\theta^{dj}(\partial_i A_d + [A_i, A_d])(\partial_j A_c + [A_j, A_c]) - \theta^{cd}\theta^{ij}\partial_i A_d \partial_j A_c \\
&= \theta^{ci}\theta^{dj}\Big(\partial_i A_d \partial_j A_c + \partial_i A_d[A_j, A_c] \\
&\qquad + [A_i, A_d]\partial_j A_c + [A_i, A_d][A_j, A_c] - \partial_d A_i \partial_j A_c\Big) \\
&= -\frac{1}{2}\theta^{cd}\theta^{ij} F_{id} F_{jc} - \frac{1}{2}\theta^{cd}\theta^{ij}[A_i, A_d][A_j, A_c] \qquad \text{(A.45)}
\end{aligned}
$$

since $\theta^{ci}\theta^{dj}[A_j, A_c]\partial_i A_d = \theta^{ci}\theta^{dj}[A_i, A_d]\partial_j A_c$ under the trace. Furthermore,

$$
[Y^a, A_a] = \theta^{ab}\partial_b A_a = -\frac{1}{2}\theta^{ab}(\partial_a A_b - \partial_b A_a) \qquad \text{(A.46)}
$$

and therefore

$$
\begin{aligned}
[X^a, A_a] &= -\frac{1}{2}\theta^{ab}(\partial_a A_b - \partial_b A_a) + \theta^{ab}[A_b, A_a] \\
&= -\frac{1}{2}\theta^{ab}(F_{ab} + [A_a, A_b]) \qquad \text{(A.47)}
\end{aligned}
$$

or

$$2\theta^{ab}[X^c, A_c]F^{ab} - [A^c, A_c]\theta^{ab}F^{ab} = -\theta^{cd}F_{cd}\theta^{ab}F^{ab} \tag{A.48}$$

hence

$$\begin{aligned}S_{SW,2} = Tr\Big(&2\theta^{ab}F^{ad}\theta^{-1}_{dc}F^{bc} + 2\theta^{ab}[A^a, A^b] + 2\theta^{ab}A_c[X^c, F^{ab}] + [A^c, A_c]\theta^{ab}F^{ab}\\ &+\frac{1}{2}\theta^{ab}\theta^{ab}A_d[\mathcal{F}^{cd}, A_c] + A_d[X^d, \theta^{ab}]A_c[X^c, \theta^{ab}]\\ &-\frac{1}{8}\theta^{ab}\theta^{ab}\theta^{cd}\theta^{ij}\Big(F_{cd}F_{ij} + 2F_{id}F_{jc} + 2F_{cd}[A_i, A_j]\Big)\Big)\end{aligned} \tag{A.49}$$

where we used

$$\begin{aligned}&Tr\theta^{ab}\theta^{ab}\theta^{cd}\theta^{ij}\Big(2[A_i, A_d][A_j, A_c] + [A_i, A_j][A_c, A_d]\Big)\\ &= -Tr\theta^{ab}\theta^{ab}\theta^{cd}\theta^{ij}\Big(2A_i[[A_j, A_c], A_d] + A_i[[A_c, A_d], A_j]\Big)\\ &= -Tr\theta^{ab}\theta^{ab}\theta^{cd}\theta^{ij}\Big(2A_i[[A_j, A_c], A_d] - A_i[[A_d, A_j], A_c] - A_i[[A_j, A_c], A_d]\Big) = 0\,.\end{aligned}$$

Together with (3.24), we obtain

$$\begin{aligned}S &= -Tr\Big(F^{ab}F^{ab} - 2F^{ab}A_c[Y^c, \theta^{ab}] + A_c[Y^c, \theta^{ab}][Y^d, \theta^{ab}]A_d + 2\theta^{ab}[A^a, A^b]\Big) + S_{SW,2}\\ &= -Tr\Big(F^{ab}F^{ab} - 2F^{ab}A_c[X^c, \theta^{ab}] + A_c[X^c, \theta^{ab}][X^d, \theta^{ab}]A_d + 2\theta^{ab}[A^a, A^b]\Big) + S_{SW,2}\,.\end{aligned}$$

Replacing $Y \to X$ which is correct to $O(\theta^4)$, we obtain

$$\begin{aligned}S &= -Tr\Big(F^{ab}F^{ab} + 2A_c[X^c, F^{ab}]\theta^{ab} + 2\theta^{ab}[X^c, A_c]F^{ab} + [X^c, \theta^{ab}]A_c[X^d, \theta^{ab}]A_d\\ &\qquad +2Tr\theta^{ab}[A^a, A^b]\Big) + S_{SW,2}\\ &= -Tr\Big(F^{ab}F^{ab} - \theta^{ab}F^{ab}\theta^{cd}F_{cd} - 2\theta^{ab}F^{ad}\theta^{-1}_{dc}F^{bc}\\ &\qquad +\frac{1}{8}\theta^{ab}\theta^{ab}\theta^{cd}\theta^{ij}\Big(F_{cd}F_{ij} + 2F_{id}F_{jc} + 2F_{cd}[A_i, A_j]\Big) - \frac{1}{2}\theta^{ab}\theta^{ab}A_d[\mathcal{F}^{cd}, A_c]\Big)\,.\end{aligned} \tag{A.50}$$

Finally we use (A.26) together with (A.48) which gives

$$Tr\theta^{ab}\theta^{ab}A_d[\mathcal{F}^{cd}, A_c] = \frac{1}{2}Tr\theta^{ab}\theta^{ab}\theta^{ij}[A_i, A_j]\theta^{cd}F_{cd} \tag{A.51}$$

and we obtain the gauge invariant action

$$\begin{aligned}S &= -Tr\Big(F^{ab}F^{ab} - \theta^{ab}F^{ab}\theta^{cd}F_{cd} - 2\theta^{ab}F^{ad}\theta^{-1}_{dc}F^{bc} + \frac{1}{8}\theta^{ab}\theta^{ab}\theta^{cd}\theta^{ij}(F_{cd}F_{ij} + 2F_{id}F_{jc})\Big)\\ &= -TrF^{ab}F^{ab} + S_{\rm NC}\end{aligned} \tag{A.52}$$

Needless to say that there should be a simpler way to obtain this.

B. Newtonian metric

We want to reproduce the metric (4.31) in terms of h_{ij} (4.20). F_{ab} is a function of (y^0, y^1, y^2, y^3) with $\eta_{ab} = (-1, 1, 1, 1)$ and has the form

$$F_{ab} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} . \tag{B.1}$$

We can assume that $\theta^{ab} = \theta \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$, which gives

$$h_{ab} = \theta^{-1} \begin{pmatrix} 2E_3 & -B_2 - E_2 & B_1 + E_1 & 0 \\ -B_2 - E_2 & -2B_3 & 0 & B_1 - E_1 \\ B_1 + E_1 & 0 & -2B_3 & B_2 - E_2 \\ 0 & B_1 - E_1 & B_2 - E_2 & -2E_3 \end{pmatrix} \tag{B.2}$$

and $\overline{g}_{ab} = \theta^{-2}(-1, -1, -1, 1)$, so that y^3 turns into the time t. Since we want the metric to be static i.e. time-independent and invariant under time reflections, we consider an electromagnetic field which is independent of y^3, $\partial_3 F_{ab} = 0$, and require

$$B_1 = E_1, \quad B_2 = E_2. \tag{B.3}$$

Then

$$h_{ab} = 2\theta^{-1} \begin{pmatrix} E_3 & -E_2 & E_1 & 0 \\ -E_2 & -B_3 & 0 & 0 \\ E_1 & 0 & -B_3 & 0 \\ 0 & 0 & 0 & -E_3 \end{pmatrix} \tag{B.4}$$

where as usual E_i and B_i can be written in the form

$$E_i = \partial_0 A_i - \partial_i A_0, \qquad B_i = \varepsilon_{ijk}\, \partial_j A_k \tag{B.5}$$

and the derivatives are w.r.t. y^a. The Bianci identities are

$$\partial_i B_i = 0, \qquad \varepsilon_{ijk}\, \partial_j E_k - \partial_0 B_i = 0\,. \tag{B.6}$$

Since we want to consider static configurations we have $\partial_3 B_3 = 0$ (recall $t = y^3$), hence

$$\partial_1 B_1 + \partial_2 B_2 = 0. \tag{B.7}$$

Now fix the gauge by setting $A_3 = 0$ (cf. axial gauge). Then

$$B_1 = -\partial_3 A_2, \quad B_2 = \partial_3 A_1, \quad E_3 = -\partial_3 A_0\,, \tag{B.8}$$

which can be solved for arbitrary B_1, B_2, E_3 satisfying the Bianci identities by

$$\begin{aligned} A_2 &= -y_3 B_1(y^0, y^1, y^2) + \tilde{A}_2(y^0, y^1, y^2), \\ A_1 &= y_3 B_2(y^0, y^1, y^2) + \tilde{A}_1(y^0, y^1, y^2), \\ A_0 &= -y_3 E_3(y^0, y^1, y^2) + \tilde{A}_0(y^0, y^1, y^2) \end{aligned} \tag{B.9}$$

with arbitrary $\tilde{A}_{0,1,2}(y^0, y^1, y^2)$. Then E_1, E_2 can be computed as

$$\begin{aligned} E_1 &= -\partial_1 \tilde{A}_0 + \partial_0 \tilde{A}_1, \\ E_2 &= -\partial_2 \tilde{A}_0 + \partial_0 \tilde{A}_2, \end{aligned} \tag{B.10}$$

where the y^3-dependent terms vanish due to the Bianci identity.

The most general B_i satisfying (B.7) can be written as

$$B_1 = \partial_2 \phi, \quad B_2 = -\partial_1 \phi \tag{B.11}$$

for any given $\phi(y^0, y^1, y^2)$. Setting $\phi = \partial_0 \varphi$ and defining $\tilde{A}_0 = 0$, $\tilde{A}_1 = \partial_2 \varphi$, $\tilde{A}_2 = -\partial_1 \varphi$ we get indeed $E_1 = B_1$, $E_2 = B_2$ and

$$B_3 = \partial_1 \tilde{A}_2 - \partial_2 \tilde{A}_1 = -\Delta_{12} \varphi. \tag{B.12}$$

E_3 is almost determined by the Bianci identity, which is solved by

$$E_3 = \partial_0 \phi = \partial_0^2 \varphi\,. \tag{B.13}$$

Now perform a change of variables $y^{a\prime} = y^a + \theta \xi^a$ with $\xi^a = 2(\phi, 0, 0, 0)$, which gives

$$h'_{ab} = 2\theta^{-1} \begin{pmatrix} -E_3 & 0 & 0 & 0 \\ 0 & -B_3 & 0 & 0 \\ 0 & 0 & -B_3 & 0 \\ 0 & 0 & 0 & -E_3 \end{pmatrix}. \tag{B.14}$$

Assuming that $O(B_3) \approx O(\partial_0 \phi)$, this describes Newtonian gravity with gravitational potential given by

$$\mathrm{U}(y^0, y^1, y^2) = \theta E_3 = \theta \partial_0^2 \varphi \tag{B.15}$$

which is arbitrary since U is arbitrary. It can therefore describe an arbitrary static mass distribution ρ by solving the Poisson equation

$$\Delta_{(3)} U = 4\pi G \rho, \tag{B.16}$$

which is expected to follow from the gravity action. For the vacuum $\rho = 0$, and $E_3 = B_3$ follows from $\Delta_{(3)} \varphi = 0$ (up to a constant), in agreement with general relativity.

C. Computation of $\eta(y)$

One way to show (3.28) is to note that

$$(\tilde{\theta} \wedge \theta)^{ijkl} = (\tilde{\theta}^{ij}\theta^{kl} - \tilde{\theta}^{il}\theta^{kj} - \tilde{\theta}^{lj}\theta^{ki}) + (\tilde{\theta}^{kl}\theta^{ij} - \tilde{\theta}^{kj}\theta^{il} - \tilde{\theta}^{ki}\theta^{lj}) \tag{C.1}$$

and to consider

$$\begin{aligned}(\theta^{-1} \wedge \theta^{-1})_{ijkl}(\tilde{\theta} \wedge \theta)^{ijkl} &= (\theta^{-1} \wedge \theta^{-1})_{ijkl}\tilde{\theta}^{ij}\theta^{kl} \\ &= (\theta^{-1}_{ij}\theta^{-1}_{kl} - \theta^{-1}_{il}\theta^{-1}_{kj} - \theta^{-1}_{lj}\theta^{-1}_{ki})\tilde{\theta}^{ij}\theta^{kl} \\ &= (\theta^{-1}_{ij}\tilde{\theta}^{ij})(\theta^{-1}_{kl}\theta^{kl}) + 2(\theta^{-1}_{il}\theta^{-1}_{jk}\tilde{\theta}^{ij}\theta^{kl}) \\ &= (d-2)\theta^{js}\theta^{sj} = (d-2)G^{ab}g_{ab}\end{aligned} \tag{C.2}$$

where $d = 4$ is the dimension of space(time). On the other hand,

$$(\theta^{-1} \wedge \theta^{-1})_{ijkl}(\theta \wedge \theta)^{ijkl} = (\theta^{-1}_{ij}\theta^{ij})(\theta^{-1}_{kl}\theta^{kl}) + 2(\theta^{-1}_{il}\theta^{-1}_{jk}\theta^{ij}\theta^{kl}) = d(d-2) \tag{C.3}$$

which together implies (3.28).

References

[1] S. Doplicher, K. Fredenhagen and J.E. Roberts, *The quantum structure of space-time at the Planck scale and quantum fields*, Commun. Math. Phys. **172** (1995) 187 [hep-th/0303037].

[2] M.R. Douglas and N.A. Nekrasov, *Noncommutative field theory*, Rev. Mod. Phys. **73** (2001) 977 [hep-th/0106048];
R.J. Szabo, *Quantum field theory on noncommutative spaces*, Phys. Rept. **378** (2003) 207 [hep-th/0109162].

[3] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, JHEP **09** (1999) 032 [hep-th/9908142].

[4] D. Kabat and W. Taylor, *Linearized supergravity from matrix theory*, Phys. Lett. **B 426** (1998) 297 [hep-th/9712185];
W. Taylor and M. Van Raamsdonk, *Supergravity currents and linearized interactions for matrix theory configurations with fermionic backgrounds*, JHEP **04** (1999) 013 [hep-th/9812239].

[5] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M-theory as a matrix model: a conjecture*, Phys. Rev. **D 55** (1997) 5112 [hep-th/9610043].

[6] D. Bigatti and L. Susskind, *Review of matrix theory*, hep-th/9712072.

[7] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, *A large-N reduced model as superstring*, Nucl. Phys. **B 498** (1997) 467 [hep-th/9612115].

[8] H. Aoki et al., *Noncommutative Yang-Mills in IIB matrix model*, Nucl. Phys. **B 565** (2000) 176 [hep-th/9908141].

[9] N. Ishibashi, S. Iso, H. Kawai and Y. Kitazawa, *String scale in noncommutative Yang-Mills*, Nucl. Phys. **B 583** (2000) 159 [hep-th/0004038].

[10] Y. Kimura and Y. Kitazawa, *Supercurrent interactions in noncommutative Yang-Mills and IIB matrix model*, *Nucl. Phys.* **B 598** (2001) 73 [hep-th/0011038].

[11] Y. Kitazawa and S. Nagaoka, *Graviton propagators in supergravity and noncommutative gauge theory*, *Phys. Rev.* **D 75** (2007) 046007 [hep-th/0611056]; *Graviton propagators on fuzzy G/H*, *JHEP* **02** (2006) 001 [hep-th/0512204].

[12] H. Kawai and M. Sato, *Perturbative vacua from IIB matrix model*, arXiv:0708.1732.

[13] M. Hanada, H. Kawai and Y. Kimura, *Describing curved spaces by matrices*, *Prog. Theor. Phys.* **114** (2006) 1295 [hep-th/0508211].

[14] Y. Okawa and H. Ooguri, *How noncommutative gauge theories couple to gravity*, *Nucl. Phys.* **B 599** (2001) 55 [hep-th/0012218].

[15] A. Dhar and Y. Kitazawa, *Non-commutative gauge theory, open Wilson lines and closed strings*, *JHEP* **08** (2001) 044 [hep-th/0106217].

[16] H. Liu and J. Michelson, *Supergravity couplings of noncommutative D-branes*, *Nucl. Phys.* **B 615** (2001) 169 [hep-th/0101016].

[17] P. Aschieri, M. Dimitrijevic, F. Meyer and J. Wess, *Noncommutative geometry and gravity*, *Class. and Quant. Grav.* **23** (2006) 1883 [hep-th/0510059];
P. Aschieri et al., *A gravity theory on noncommutative spaces*, *Class. and Quant. Grav.* **22** (2005) 3511 [hep-th/0504183].

[18] V.P. Nair, *The Chern-Simons one-form and gravity on a fuzzy space*, *Nucl. Phys.* **B 750** (2006) 321 [hep-th/0605008].

[19] X. Calmet and A. Kobakhidze, *Noncommutative general relativity*, *Phys. Rev.* **D 72** (2005) 045010 [hep-th/0506157];
A. Kobakhidze, *Theta-twisted gravity*, hep-th/0603132.

[20] J. Madore, *An introduction to noncommutative differential geometry and physical applications*, *Lond. Math. Soc. Lect. Note Ser.* **257** (2000) 1;
J. Madore and J. Mourad, *Quantum space-time and classical gravity*, *J. Math. Phys.* **39** (1998) 423 [gr-qc/9607060].

[21] M.A. Cardella and D. Zanon, *Noncommutative deformation of four dimensional Einstein gravity*, *Class. and Quant. Grav.* **20** (2003) L95 [hep-th/0212071].

[22] E. Langmann and R.J. Szabo, *Teleparallel gravity and dimensional reductions of noncommutative gauge theory*, *Phys. Rev.* **D 64** (2001) 104019 [hep-th/0105094].

[23] R.J. Szabo, *Symmetry, gravity and noncommutativity*, *Class. and Quant. Grav.* **23** (2006) R199 [hep-th/0606233].

[24] S. Kurkcuoglu and C. Saemann, *Drinfeld twist and general relativity with fuzzy spaces*, *Class. and Quant. Grav.* **24** (2007) 291 [hep-th/0606197].

[25] P. Mukherjee and A. Saha, *Comment on the first order noncommutative correction to gravity*, *Phys. Rev.* **D 74** (2006) 027702 [hep-th/0605287];
R. Banerjee, P. Mukherjee and S. Samanta, *Lie algebraic noncommutative gravity*, *Phys. Rev.* **D 75** (2007) 125020 [hep-th/0703128].

[26] W. Behr, F. Meyer and H. Steinacker, *Gauge theory on fuzzy $S^2 \times S^2$ and regularization on noncommutative R^4*, *JHEP* **07** (2005) 040 [hep-th/0503041];
H. Grosse and H. Steinacker, *Finite gauge theory on fuzzy CP^2*, *Nucl. Phys.* **B 707** (2005) 145 [hep-th/0407089].

[27] Y. Kitazawa, *Matrix models in homogeneous spaces*, *Nucl. Phys.* **B 642** (2002) 210 [hep-th/0207115].

[28] S. Minwalla, M. Van Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, *JHEP* **02** (2000) 020 [hep-th/9912072].

[29] M. Hayakawa, *Perturbative analysis on infrared and ultraviolet aspects of noncommutative QED on R^4*, hep-th/9912167.

[30] A. Matusis, L. Susskind and N. Toumbas, *The IR/UV connection in the non-commutative gauge theories*, *JHEP* **12** (2000) 002 [hep-th/0002075].

[31] V.O. Rivelles, *Noncommutative field theories and gravity*, *Phys. Lett.* **B 558** (2003) 191 [hep-th/0212262].

[32] H.S. Yang, *Exact Seiberg-Witten map and induced gravity from noncommutativity*, *Mod. Phys. Lett.* **A 21** (2006) 2637 [hep-th/0402002]; *On the correspondence between noncommuative field theory and gravity*, *Mod. Phys. Lett.* **A 22** (2007) 1119 [hep-th/0612231]; *Noncommutative electromagnetism as a large-N gauge theory*, arXiv:0704.0929.

[33] R. Banerjee and H.S. Yang, *Exact Seiberg-Witten map, induced gravity and topological invariants in noncommutative field theories*, *Nucl. Phys.* **B 708** (2005) 434 [hep-th/0404064].

[34] A.D. Sakharov, *Vacuum quantum fluctuations in curved space and the theory of gravitation*, *Sov. Phys. Dokl.* **12** (1968) 1040 [*Dokl. Akad. Nauk Ser. Fiz.* **177** (1967) 70] [*Sov. Phys. Usp.* **34** (1991) 394].

[35] J. Madore, S. Schraml, P. Schupp and J. Wess, *Gauge theory on noncommutative spaces*, *Eur. Phys. J.* **C 16** (2000) 161 [hep-th/0001203].

[36] P. Aschieri, T. Grammatikopoulos, H. Steinacker and G. Zoupanos, *Dynamical generation of fuzzy extra dimensions, dimensional reduction and symmetry breaking*, *JHEP* **09** (2006) 026 [hep-th/0606021];
H. Steinacker and G. Zoupanos, *Fermions on spontaneously generated spherical extra dimensions*, *JHEP* **09** (2007) 017 [arXiv:0706.0398].

[37] M. Kontsevich, *Deformation quantization of Poisson manifolds, I*, *Lett. Math. Phys.* **66** (2003) 157 [q-alg/9709040].

[38] B. Jurco and P. Schupp, *Noncommutative Yang-Mills from equivalence of star products*, *Eur. Phys. J.* **C 14** (2000) 367 [hep-th/0001032];
B. Jurco, S. Schraml, P. Schupp and J. Wess, *Enveloping algebra valued gauge transformations for non- abelian gauge groups on non-commutative spaces*, *Eur. Phys. J.* **C 17** (2000) 521 [hep-th/0006246].

[39] E. Alvarez, *Can one tell Einstein's unimodular theory from Einstein's general relativity?*, *JHEP* **03** (2005) 002 [hep-th/0501146].

[40] M. Henneaux and C. Teitelboim, *The cosmological constant and general covariance*, *Phys. Lett.* **B 222** (1989) 195.

[41] A. Einstein, *Do gravitational fields play an essential part in the structure of the elementary particles of matter?*, *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* **1919** (1919) 433.

[42] M. Visser, *Sakharov's induced gravity: a modern perspective*, *Mod. Phys. Lett.* **A 17** (2002) 977 [gr-qc/0204062].

[43] P.B. Gilkey, *Invariance theory, the heat equation and the Atiyah-Singer index theorem*, Wilmington, Publish or Perish (1984).

[44] A.H. Chamseddine and A. Connes, *The spectral action principle*, *Commun. Math. Phys.* **186** (1997) 731 [hep-th/9606001].

[45] V. Gayral and B. Iochum, *The spectral action for Moyal planes*, *J. Math. Phys.* **46** (2005) 043503 [hep-th/0402147].

[46] H. Grosse and M. Wohlgenannt, *Induced gauge theory on a noncommutative space*, *Eur. Phys. J.* **C 52** (2007) 435 [hep-th/0703169].

[47] D.N. Blaschke, H. Grosse and M. Schweda, *Non-commutative* U(1) *gauge theory on R^4 with oscillator term*, *Europhys. Lett.* **79** (2007) 61002 [arXiv:0705.4205].

[48] H. Steinacker, *Quantized gauge theory on the fuzzy sphere as random matrix model*, *Nucl. Phys.* **B 679** (2004) 66 [hep-th/0307075].

[49] D. Dou and B. Ydri, *Topology change from quantum instability of gauge theory on fuzzy $CP(2)$*, *Nucl. Phys.* **B 771** (2007) 167 [hep-th/0701160].

[50] T. Azuma, S. Bal, K. Nagao and J. Nishimura, *Dynamical aspects of the fuzzy $CP(2)$ in the large-N reduced model with a cubic term*, *JHEP* **05** (2006) 061 [hep-th/0405277];
T. Azuma, S. Bal and J. Nishimura, *Dynamical generation of gauge groups in the massive Yang-Mills-Chern-Simons matrix model*, *Phys. Rev.* **D 72** (2005) 066005 [hep-th/0504217].

[51] F. Lizzi, R.J. Szabo and A. Zampini, *Geometry of the gauge algebra in noncommutative Yang-Mills theory*, *JHEP* **08** (2001) 032 [hep-th/0107115].

[52] T. Azuma, S. Iso, H. Kawai and Y. Ohwashi, *Supermatrix models*, *Nucl. Phys.* **B 610** (2001) 251 [hep-th/0102168].